

1. In an unweighted, undirected graph G , we have seen that calculating the length of the shortest path from a node u to another node v can be accomplished efficiently by performing a breadth-first search.

Yet it is often the case that the shortest path from u to v is not unique. In this problem, we are interested in counting the number of distinct shortest paths which exist between nodes s and t . Design as efficient of an algorithm as possible for calculating the number of such paths; note that your algorithm does not need to list all of those paths, just the number of them.

2. In internet routing, there are delays both on lines (edges) and on routers (vertices). The motivates the following question on directed graphs:

Suppose that in addition to edge lengths $\ell(u \rightarrow v)$ for every $u \rightarrow v \in E$, there is also a vertex cost $c(v)$ for every $v \in V$. Define the cost of a path to be equal to the sum of its edge lengths, plus the sum of all vertices in the path, including its endpoints. (In this problem I'll index vertices as $1, 2, \dots, |V|$ for simplicity, so each vertex is just a number, but if you prefer you can think of vertex i as v_i .)

Give an efficient algorithm which, given a directed graph $G = (V, E)$, set of positive edge lengths $\ell(e)$ and positive vertex costs $c(v)$, and a starting vertex $s \in V$, computes an array $cost[1 \dots |V|]$ such that for every $i \in V$, $cost[i]$ stores the least cost path from s to i .

(Note: to get you started, please notice that $cost[s] = c(s)$.)

3. Now that finals are ending, it's time for games! A friend of yours has created a game based on J.R.R. Tolkien's *The Lord of the Rings* (one of my favorites). The game takes place in Middle Earth. In this game, there's a set of hobbits, humans, dwarves and elves (aka "the good guys") who are under attack in their homes, and need to escape to a set of safe points where they cannot be harmed.
- For the m safe points $1, \dots, m$, the i^{th} safe point can hold at most $S[i]$ of the good guys. (You can assume the safe points begin empty.)
 - Meanwhile, for the set of n homes $1, \dots, n$, each home region j holds $H[j]$ good guys.
 - Finally, for each home region j , we have a list of the safe points that can be reached in at most 3 days travel. (You may assume this is stored as an array $R[1..m][1..n]$, where $R[i][j]$ is true if home region j can reach safe region i within 3 days.)

Your job is to save as many of the good guys as possible! Fresh from your algorithms final, you recognize something familiar about this problem...

Given the input as described above, describe an algorithm to move as many of the good guys from their home locations to safe points in three days time, while not overcrowding any stronghold. Your algorithm should be as fast as possible.

4. A university admissions office is planning its outreach campaign for the next recruiting cycle. The office wants to use a combination of email campaigns, campus visits, and social media ads to maximize the expected number of prospective students reached, subject to budget and staff constraints.

The outreach office has at most \$500,000 to spend overall. The following table gives the estimated reach, cost, and maximum number of uses for each type of outreach.

type	students reached per use	cost per use	max uses
email campaign	8,000	5,000	40
campus visit	3,000	20,000	15
social media ad	12,000	25,000	20

In addition, campus visits and social media ads both require work from the same outreach team, which can support at most 25 total uses of these two types.

- (a) Design a linear program in canonical form to determine the best combination of outreach activities. You do not need to solve the LP, and fractional uses are allowed.
- (b) Give the dual formulation of the linear program you wrote in part (a).

5. Over break, you are working part time as a security consultant currently at that well known big company, Elgoog. You have just discovered something awful: a huge number of computers in your network are infected with malware that must have come from users visiting unsafe websites. Luckily, you are able to build a log file for each computer that lists all websites it has visited since the last time they were scanned for malware. Unfortunately, as you look these files over, you notice that there is not a single website that every machine has visited. You conclude, therefore, that there must be a number of websites that are able to inject this malware, and the most likely candidate list would be in a smallest collection of sites such that each computer visited at least one of those sites.

Show that the decision version of this problem (i.e. can you find a set of size k of such websites, so that every computer logged as visited at least one of the k websites) is NP-Complete.

6. Let $f : \{1, 2, \dots, n\} \rightarrow \mathbb{R}$ satisfy:

- $f(1) < 0$
- $f(n) > 0$

You are writing an algorithm to find an index i such that $f(i) < 0$ and $f(i + 1) > 0$. Your algorithm may query values of $f(i)$.

(a) Give an algorithm to find such an index i , using as few queries as possible.

(b) Prove a lower bound for the number of queries necessary.

7. Given a multiset $A \subseteq \{0, 1, \dots, 10n\}$ of size n , compute the number of triples (i, j, k) such that $a_i + a_j = a_k$ in $O(n \log n)$ time. [Hint: you can do this using FFTs.]

You may assume the following problems are NP-Hard:

(Note that I'm giving the decision versions here, which all have a yes/no answer.)

- CIRCUICTSAT: Given a boolean circuit, are there any input values that make the circuit output True?
- 3SAT: Given a boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?
- INDEPENDENTSET: Given an undirected graph G and a positive integer k , is there a subset of vertices in G of size k that have no edges among them?
- KCLIQUE: Given an undirected graph G and a positive integer k , is there a clique in G of size k ?
- VERTEXCOVER: Given an undirected graph G and a positive integer k , is there a subset of vertices of size k that touch every edge in G ?
- HITTINGSET: Given a collection of subsets S_1, S_2, \dots, S_m of a set S , is there a subset of S of size k that intersects every subset S_i ?
- HAMILTONIANCYCLE: Given a graph G , is there a cycle in G that visits every vertex exactly once?
- HAMILTONIANPATH: Given a graph G , is there a path in G that visits every vertex exactly once?
- TRAVELINGSALESMAN: Given a graph G with weighted edges and a value k , is there a Hamiltonian path/cycle in G of total weight $\leq k$?
- SUBSETSUM: Given a set X of positive integers and an integer k , does X have a subset whose elements sum to k ?
- PARTITION: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?
- 3COLORABLE: Given a graph, does it have a 3-coloring?

(scratch paper)

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