Algorithms – Spring '25

Review for HW0

Last time:

- New classroom!
- Overview of syllabus:

  2 main resources:

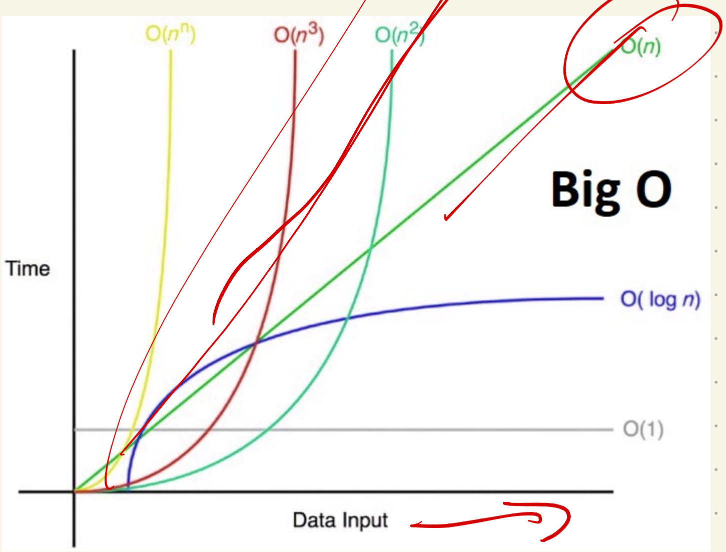
  - My course page
    ↳ Lots of typos/links fixed, so thanks for emails!
  - Canvas
    ↳ links to Gradescope & Perusall

- HW0: due next Wed.

Some background reminders

① Big-O: ignore constants!
Also, smaller terms "disappear":

- $5 \cdot 2^n + 6n^3 + 3$
  $= O(2^n)$
- $\frac{5}{3}n^2 - 33n + 1$
  $= O(n^2)$
- 6n
  $1{,}634\,n^2 + 10n + 1{,}000{,}000$
  5n $= O(n^2)$



More formally: $f(n)$ is $O(g(n))$ ← set of fcns

if $\exists c > 0$ & $N_0 > 0$ such that $\forall n > N_0$,

$$f(n) \le c \cdot g(n)$$

$\Rightarrow O(g) \Rightarrow \in \{O(g)\}$

<u>Prove</u>: $5n^2 + 11n - 6$ is $O(n^2)$:

Find c and N, & prove $\forall n > N_0$, the inequality: Fix $N_0 = 1$.

$$5n^2 + 11n - 6 \le 5n^2 + 11n^2 + 6n^2$$

$$= 22n^3$$

Set $c = 22$, $\forall n > 1$ ($\leftarrow N_0$),

& $5n^2 + 11n - 6 \le 22n^2$ ∎

② logarithms: useful identities!

Find it in your discrete math reference, ie →

=exp I raise b to in order to get xy

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

$\log_b(x^y) = y\log_b(x)$

$\log_b(\sqrt[y]{x}) = \frac{\log_b(x)}{y}$

$\log_b a$:= number I raise b to in order to get a

$\log_2(2^8) = 8$ $\quad \log_2 64 = 5$

Why? logarithms are all about exponents!

What is $2^a \cdot 2^b$?

$(2^a)\cdot(2^b) = 2^{a+b}$

$\frac{5^6}{5^2} = 5^{6/2}$

$2^{ab} = (2^a)^b = (2^b)^a$

Another:

$$\log_a b = \left.\frac{\log_x b}{\log_x a}\right] \text{ with } \cancel{\text{any}} \text{ base } x$$

Question: Is $\log_{10} n$ big-O of $\log_2 n$?

$$\log_{10} n = \frac{\log_2 n}{\log_2 10}$$

use rule

shorthand $\log_2 n$

$$= \underbrace{\left(\frac{1}{\log_2 10}\right)^{=c}}_{\text{constant between 3 \& 4}} \cdot \lg n \leq c \cdot \log_2 n$$

$$= O(\lg n)$$

Ex: what about

$\log_a(a^x)$?

$\Rightarrow x$

or : $\log_2(n^2)$

$= 2 \cdot \log_2 n$

$(a^b)^c = a^{bc}$

Not same as:

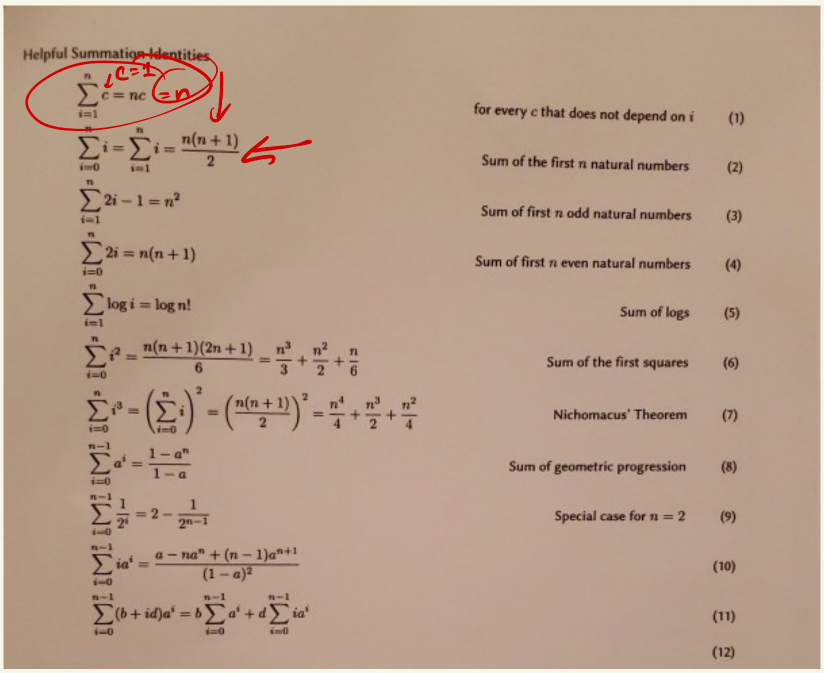
$(\log_2 n)^2 = \lg^2 n$

③ Summations:

again, your discrete math book has a table.

Find it. Love it.

Ex:



Helpful Summation Identities

$$\sum_{i=1}^{n} c = nc \quad \text{for every } c \text{ that does not depend on } i \qquad (1)$$

$c=1$, $=n$

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{Sum of the first } n \text{ natural numbers} \qquad (2)$$

$$\sum_{i=1}^{n} 2i-1 = n^2 \quad \text{Sum of first } n \text{ odd natural numbers} \qquad (3)$$

$$\sum_{i=0}^{n} 2i = n(n+1) \quad \text{Sum of first } n \text{ even natural numbers} \qquad (4)$$

$$\sum_{i=1}^{n} \log i = \log n! \quad \text{Sum of logs} \qquad (5)$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \quad \text{Sum of the first squares} \qquad (6)$$

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \quad \text{Nichomacus' Theorem} \qquad (7)$$

$$\sum_{i=0}^{n-1} a^i = \frac{1-a^n}{1-a} \quad \text{Sum of geometric progression} \qquad (8)$$

$$\sum_{i=0}^{n-1} \frac{1}{2^i} = 2 - \frac{1}{2^{n-1}} \quad \text{Special case for } n=2 \qquad (9)$$

$$\sum_{i=0}^{n-1} ia^i = \frac{a - na^n + (n-1)a^{n+1}}{(1-a)^2} \qquad (10)$$

$$\sum_{i=0}^{n-1} (b+id)a^i = b\sum_{i=0}^{n-1} a^i + d\sum_{i=0}^{n-1} ia^i \qquad (11)$$

(12)

Notation: $\sum_{i=1}^{n} f(i) = f(1) + f(2) + \dots + f(n)$

upper $\rightarrow n$

lower $\rightarrow i=1$

Ex: $\sum_{i=1}^{n} 1 = \overbrace{1 + 1 + \dots + 1}^{n \text{ times}} = n$, or $\sum_{i=1}^{n} i = 1 + 2 + 3 + \dots + n$

$f(i) = i$

$= (n+1)\frac{n}{2}$

$= \frac{n(n+1)}{2}$

$$\sum_{i=1}^{n}(2i+1)$$

$$= \cancel{3+5+7+\cdots+(2n+1)}$$

$$= \sum_{i=1}^{n} 2i + \underbrace{\sum_{i=1}^{n} 1}_{= n}$$

$$= n + \underbrace{\sum_{i=1}^{n} 2i}_{}$$

$$2\cdot 1 + 2\cdot 2 + 2\cdot 3 + \cdots 2\cdot n$$
$$= (2)(1+\cdots+n)$$

$$= n + 2\left(\sum_{i=1}^{n} i\right)$$

$$= n + \cancel{2}\left(\frac{n(n+1)}{\cancel{2}}\right) = 2n + n^2$$

$$= O(n^2)$$

4 Induction:

There is a template!

Base case: Prove statement for small value

Ind hypothesis: Assume true for values $\leq k$.

Ind. step: Prove true for next value $k+1$

Think of this as "automating" a proof.

- $P(1)$
- $\forall k \geq 1$, $\overbrace{\text{if } P(k)}^{IH}$ then $\underbrace{P(k+1)}_{IS}$

Learn it, use it, + love it!

Example:

EXAMPLE 3 Use mathematical induction to show that

$$\text{LHS} \rightarrow \underline{1 + 2 + 2^2 + \cdots + 2^n} = \underline{2^{n+1} - 1} \leftarrow \text{RHS}$$

Base case:

Fix $n=1$:

$\text{LHS} = 1 + 2^1 = 3$

$\text{RHS} = 2^{1+1} - 1 = 4 - 1 = 3$ ✓

they are equal

IH: Assume true for any value $\leq k-1$: Assume $P(k-1)$

$$1 + 2 + \cdots + 2^{k-1} = 2^k - 1$$

IS: Prove true for k:

Consider $\underbrace{1 + 2 + 2^2 + \cdots + 2^{k-1}}_{\text{By IH, } = 2^k - 1} + \underline{2^k}$

LHS

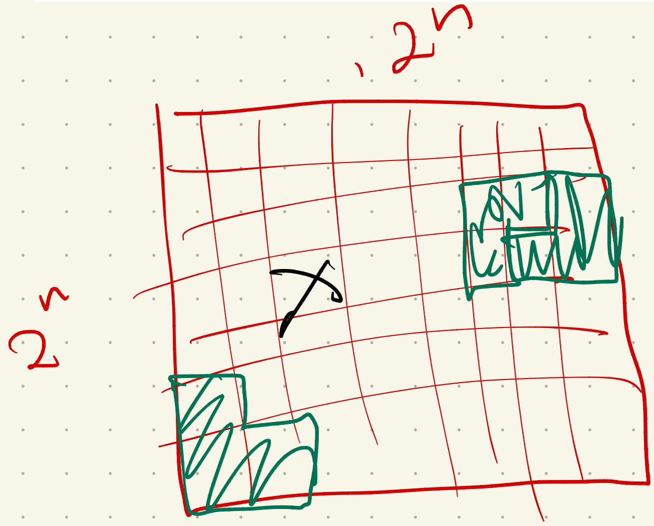
so $= (2^k - 1) + 2^k = 2^k + 2^k - 1$

$= 2^1 \cdot 2^k - 1 = 2^{1+k} - 1 = 2^{k+1} - 1$

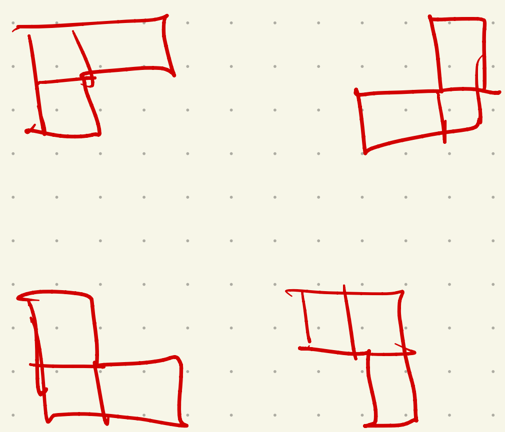
∎

"Structural" induction:

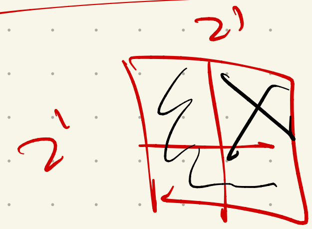
Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, where these pieces cover three squares at a time, as shown in Figure 4.



triominos

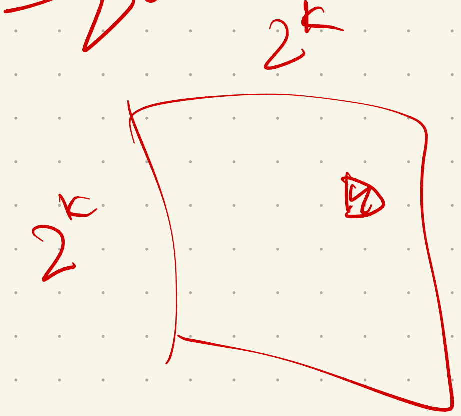


Base case: $2^1 \times 2^1$ board



No matter which square is removed, I can cover the other 3

IH: Assume I can tile any $2^k \times 2^k$ board with any 1 square removed



IS: Show I can do a $2^{k+1} \times 2^{k+1}$ board: