Algorithms - Spring '25

Directed graphs

Recap

- Some left black sweatshirt in my office
- Oral grading
- Exam next

Chapter 6:

All about directed graphs!

First, though, some things to recall: graph traversals.

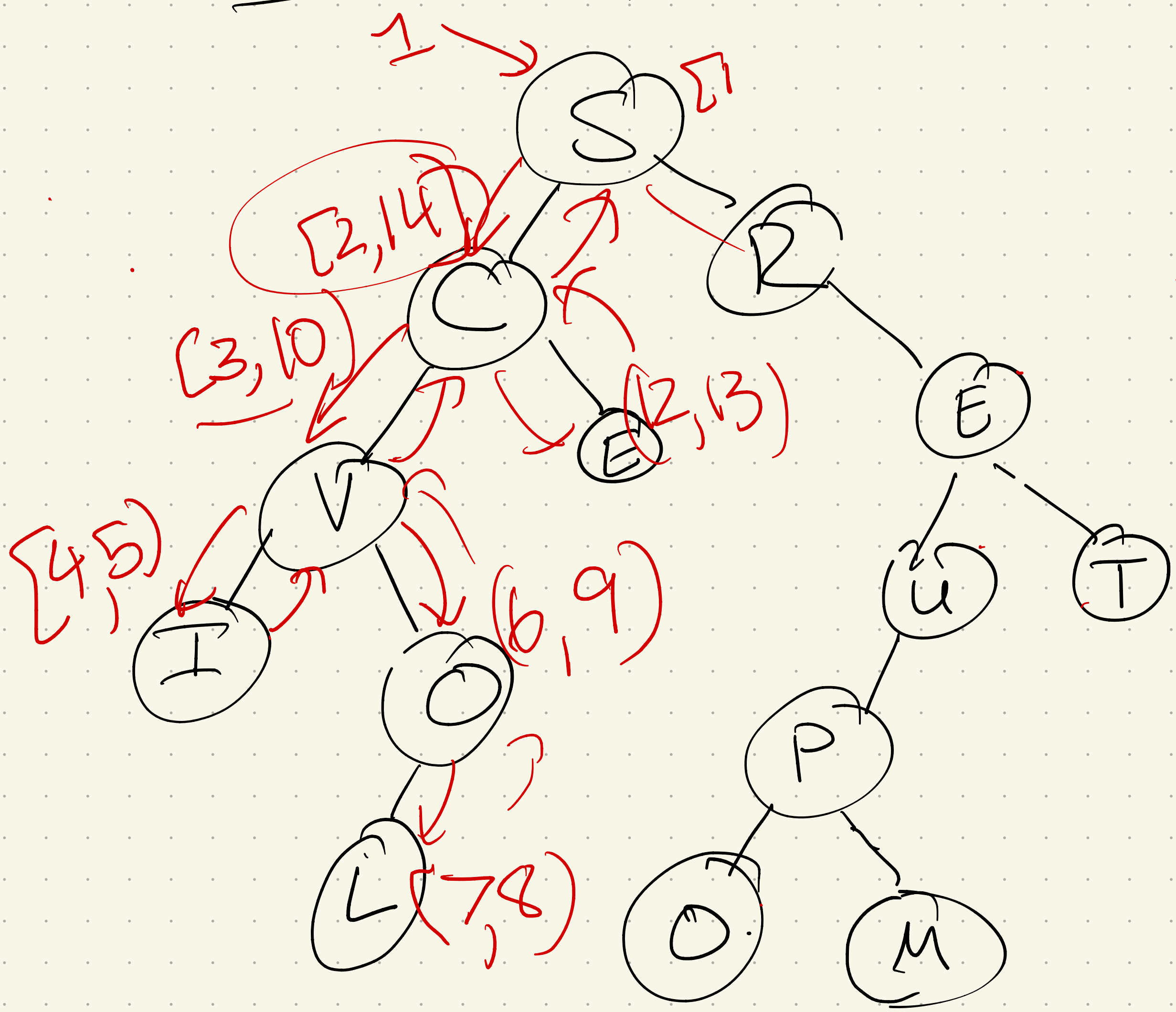
- Pre-order: visit v
  visit children
- Post-order: visit all children
  visit v

⇓

We can use these on more general graphs!

Searching & directed graphs:

Recall: post order traversal



- imagine a "clock" incrementing each time an edge is traversed!

General graphs:

DFS(a) :
DFS(b)
DF(a)

DFSALL(G):
  $clock \leftarrow 0$
  for all vertices v
    unmark v
  for all vertices v
    if v is unmarked
      $clock \leftarrow$ DFS(v, $clock$)

DFS(v, $clock$):
  mark v
  $clock \leftarrow clock + 1$; $v.pre \leftarrow clock$
  for each edge $v \to w$
    if w is unmarked
      $w.parent \leftarrow v$
      $clock \leftarrow$ DFS(w, $clock$)
  $clock \leftarrow clock + 1$; $v.post \leftarrow clock$
  return $clock$

a→b

a to be b's parent in tree

Parent

a.pre

b.pre

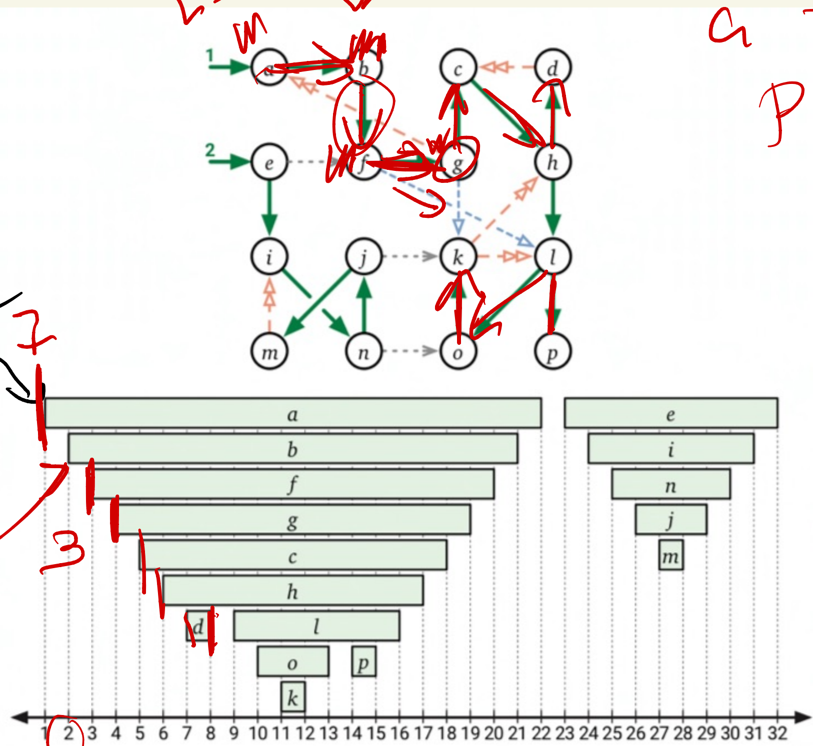


Figure 6.4. A depth-first forest of a directed graph, and the corresponding active intervals of its vertices, defining the preordering *abfgchdlokpeinjm* and the postordering *dkoplhcgfbamjnie*. Forest edges are solid; dashed edges are explained in Figure 6.5.

clock: ~~0~~ ~~1~~ ~~2~~ 3

v=a

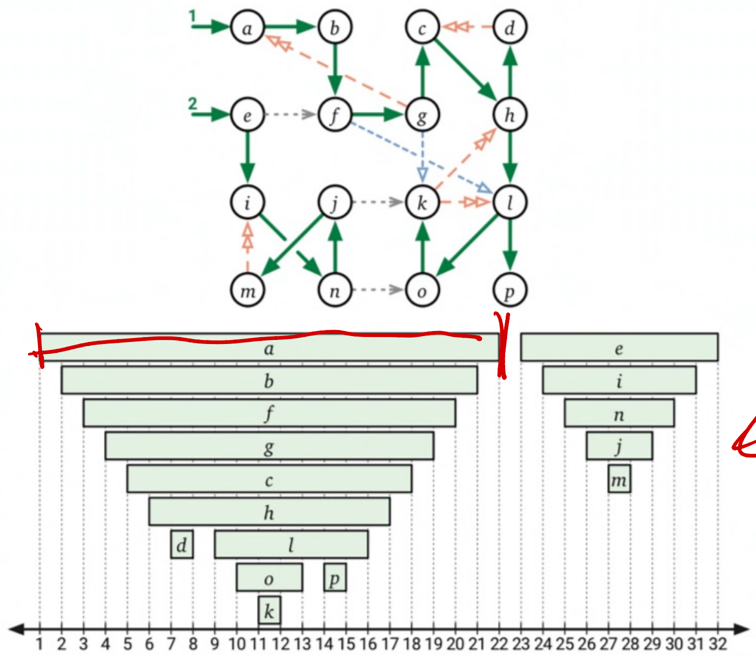
a is unmarked: DFS(a, clock) 0

DFS(b, 1)

DFS(f, 2)

DFS(g

Result:



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So: in DFS, this "lifespan represents how long a vertex is on the stack.

(we saw recursive, not stack)

Notation:

[v.pre, v.post]

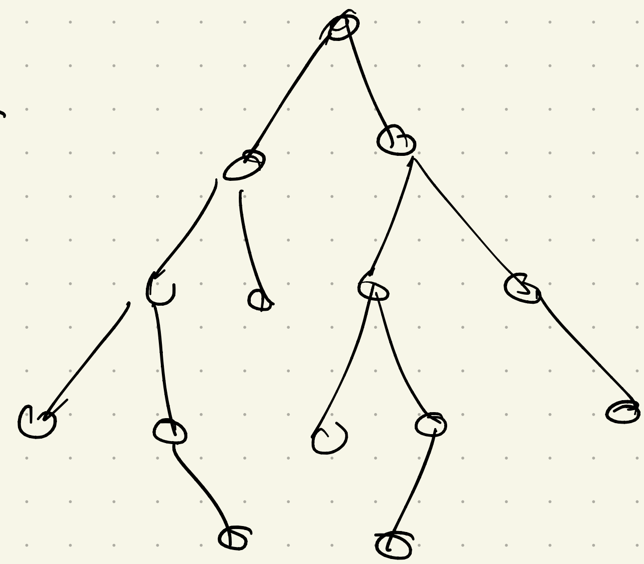
preorder visit time

postorder visit time

Note: In general graphs, post order traversal is not unique!

It was in BSTs:

left,
right,
self



In graphs:
Just use adj. list order

a → b → e
b → a → e → f → c
c → b → f → g → d
d → c → g
e → a → b → f → g → h
f → e → b → c → g
g → i → e → f → c → d
h → e
i → g
j → l → m → k
k → j → l
l → m → j → k
m → l → j

```
DFSAll(G):
    clock ← 0
    for all vertices v
        unmark v
    for all vertices v
        if v is unmarked
            clock ← DFS(v, clock)
```

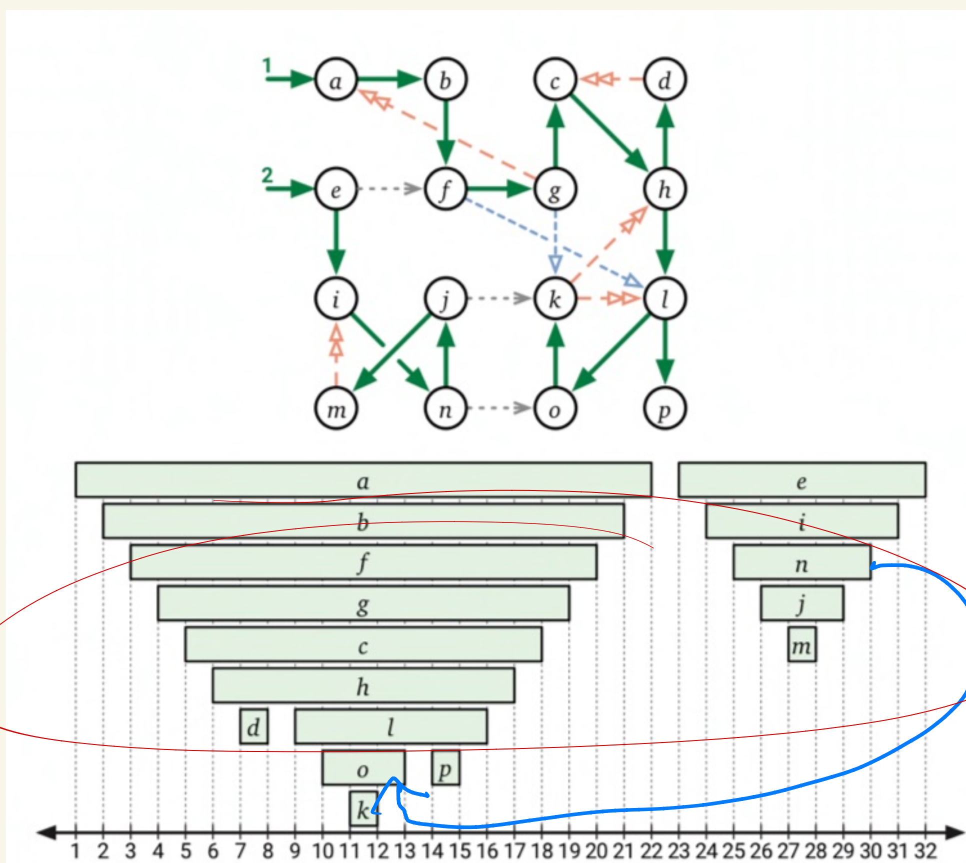
```
DFS(v, clock):
    mark v
    clock ← clock + 1; v.pre ← clock
    for each edge v→w
        if w is unmarked
            w.parent ← v
            clock ← DFS(w, clock)
    clock ← clock + 1; v.post ← clock
    return clock
```

Dfn: - tree edge: (green)
u→v
DFS(u) calls DFS(v) & v not visited befor

- forward edge: (blue)
v's interval contained in u's, v not visited yet, u doesn't call v

- back edge: (red)
v already active when u is called

- cross edge: (grey)
v is done before u is called



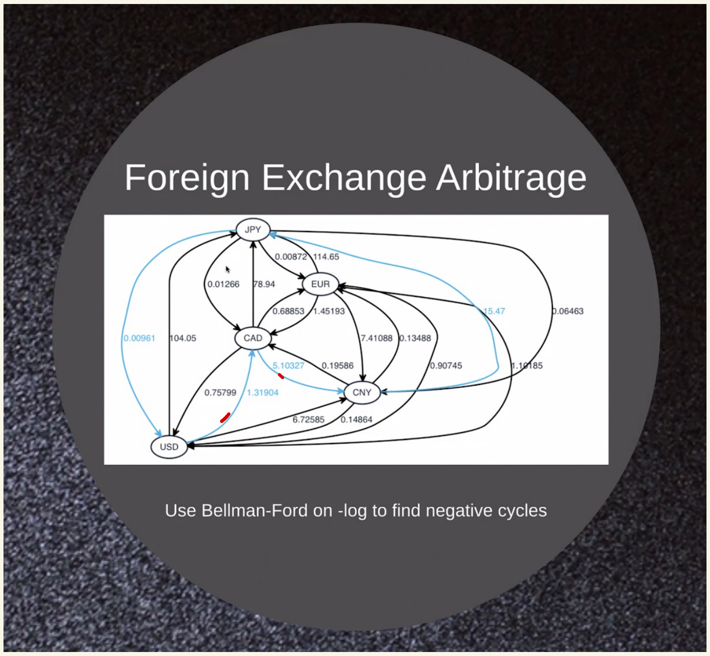
Finding cycles

In general, cycles tend to be important.

Sometimes bad:

- topological ordering in a DAG (see next slides)
- longer run time
  ↳ see Dyn. Pro.

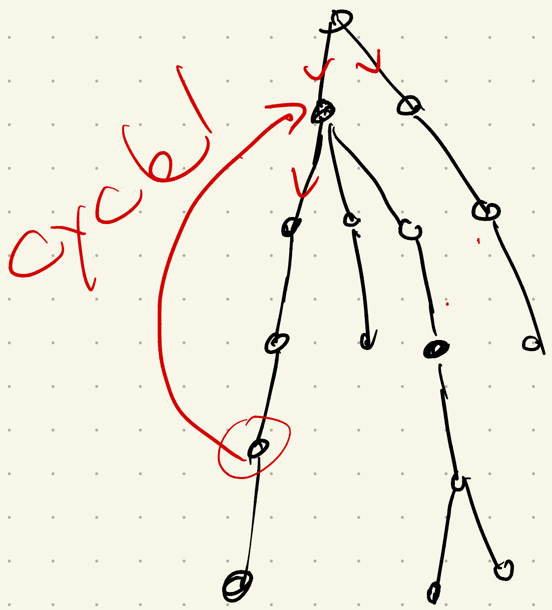
Sometimes good:



(taken from a talk I saw, by a person who works in high frequency trading)

Suppose $u \to v$, $u.post < v.post$:
u was removed from "active" stack before v.

(and not a cross edge)
Where can u be?



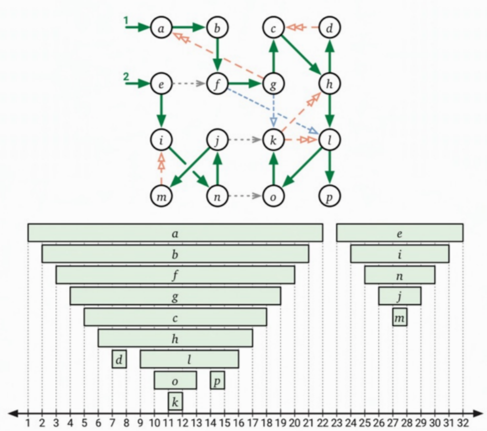


Figure 6.4. A depth-first forest of a directed graph, and the corresponding active intervals of its vertices, defining the preordering *abfgchdlokpeinjm* and the postordering *dkoplhcgfbamjnie*. Forest edges are solid; dashed edges are explained in Figure 6.5.

We can use this!
To detect cycles, + order
(if not present).

Topological ordering: Why?

Track dependencies:

- class prereqs
- compilers + #includes
- ordering evaluations of cells in a spreadsheet
- data analysis pipelines

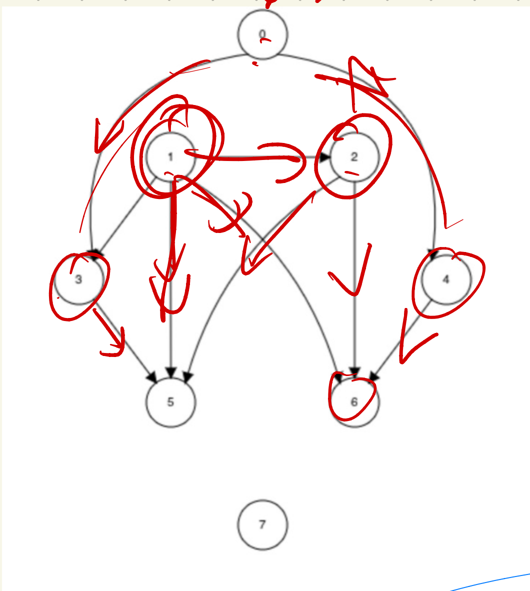
⋮

Often, in all these settings, the goal is to find a processing order that works

↳ lays out dependencies so precursor is evaluated first!

A simple example

Start at 0



Post-ordering:

visit children
↳ then mark self

Here:

$$\overset{1}{V_5} \leftarrow \overset{2}{V_3} \quad \overset{3}{V_6} \leftarrow \overset{4}{V_4} \leftarrow \overset{5}{V_0} \quad \overset{6}{V_2} \leftarrow \overset{7}{V_1} \quad \overset{8}{V_7}$$

Note: this puts a vertex "after" anything it can reach!

So, to get a post-ordering: reverse it!

Why does it work? (b/c DFS)

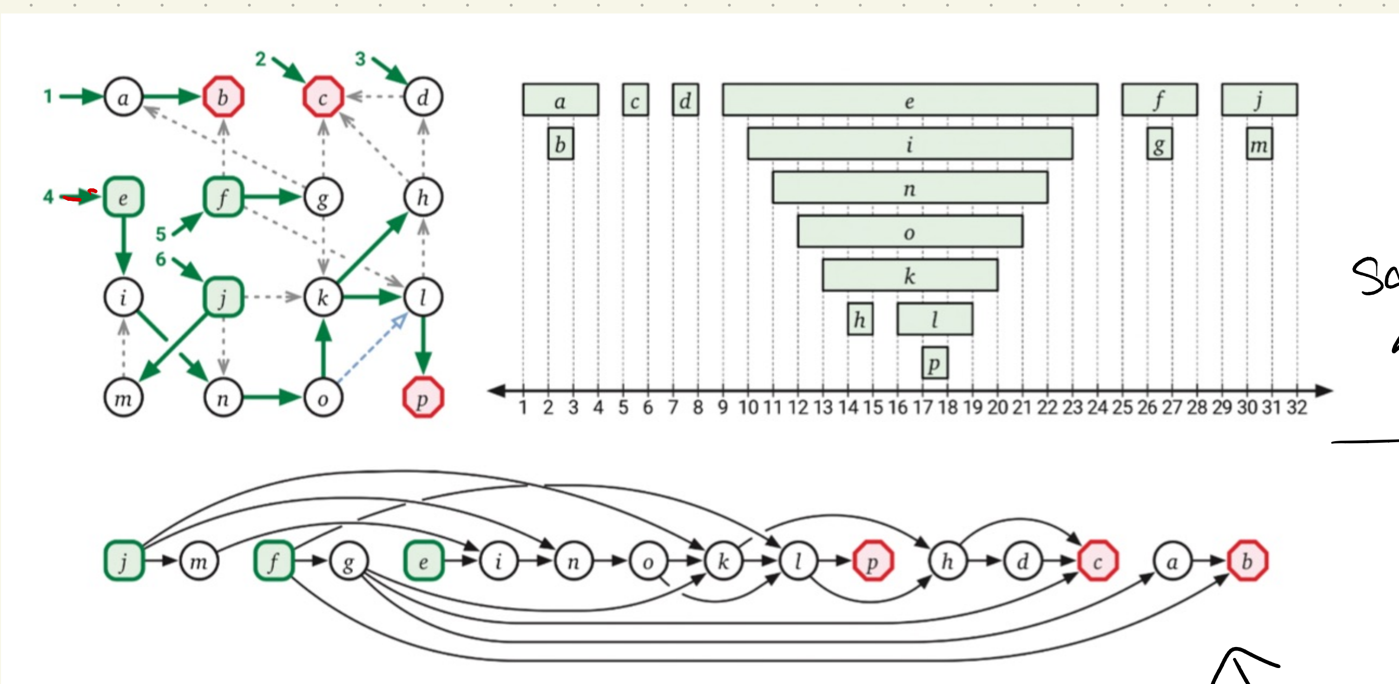
Top sort DFS: making it more precise

```
TopologicalSort(G):
    for all vertices v
        v.status ← New
    clock ← V
    for all vertices v
        if v.status = New
            clock ← TopSortDFS(v, clock)
    return S[1..V]
```

```
TopSortDFS(v, clock):
    v.status ← Active
    for each edge v→w
        if w.status = New
            clock ← TopSortDFS(v, clock)
        else if w.status = Active
            fail gracefully
    v.status ← Finished
    S[clock] ← v
    clock ← clock − 1
    return clock
```

Figure 6.9. Explicit topological sort

Unpacking his figure:



Same

Figure 6.8. Reversed postordering of the dag from Figure 6.6.

New

Memoization + DP

Nice connection!

If the graph is a DAG, can do dynamic programming on it.

Why?

Think of the recurrences:

$$T(v) = \max_{\substack{\text{(predecessors} \\ \text{or successors } u \\ \text{of } v)}} \left\{ \begin{array}{c} T(u) \\ \text{lookup} + \\ \text{calculation} \end{array} \right\}$$

When will the algorithm get stuck?

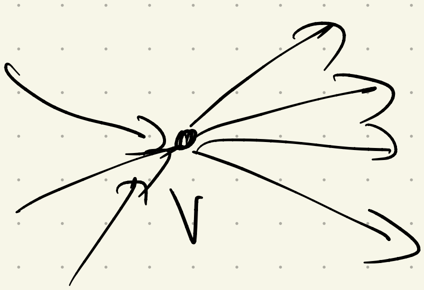
Example: longest path in a DAG.

Usually → very hard.

Think backtracking for a moment, & fix a "target" vertex t.

Let $LLP(v)$ = longest path from v to t

$$= \max \left\{ \vphantom{\int} \right.$$



Using this recursion:
"memoize" the value LLP:
Add a field to the vertex
& store it.
(Initially, =)

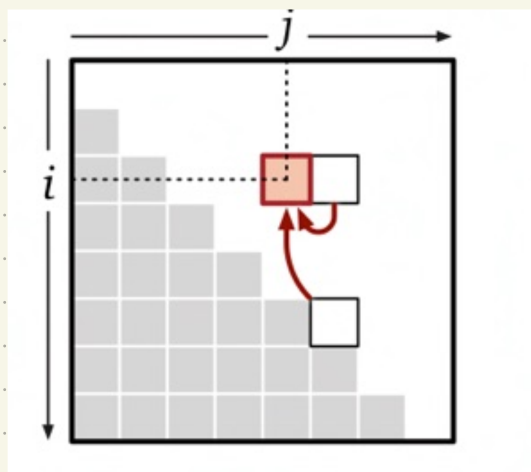
<u>GetLongest(v)</u>:
if $v = t$:

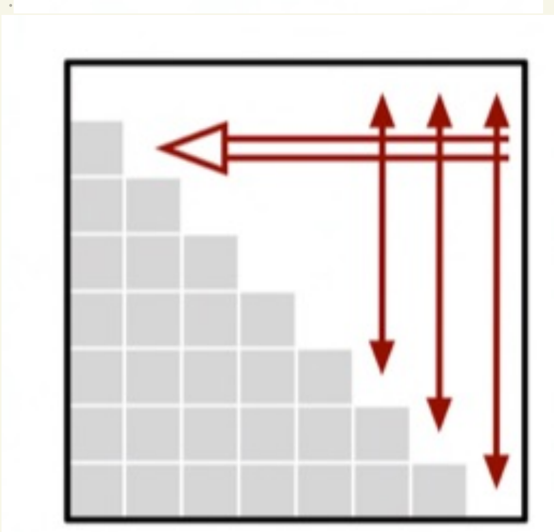
otherwise:

In principle, every DP we saw is working on a dependency graph of subproblems!

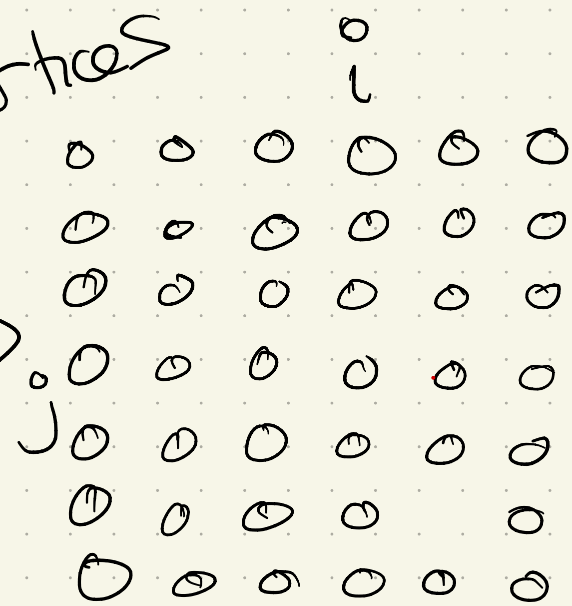
<u>Recall</u>: Longest Inc. Subsequence

$$
LISbigger(i,j) = \begin{cases} 0 & \text{if } j > n \\ LISbigger(i, j+1) & \text{if } A[i] \geq A[j] \\ \max \begin{Bmatrix} LISbigger(i, j+1) \\ 1 + LISbigger(j, j+1) \end{Bmatrix} & \text{otherwise} \end{cases}
$$





vertices



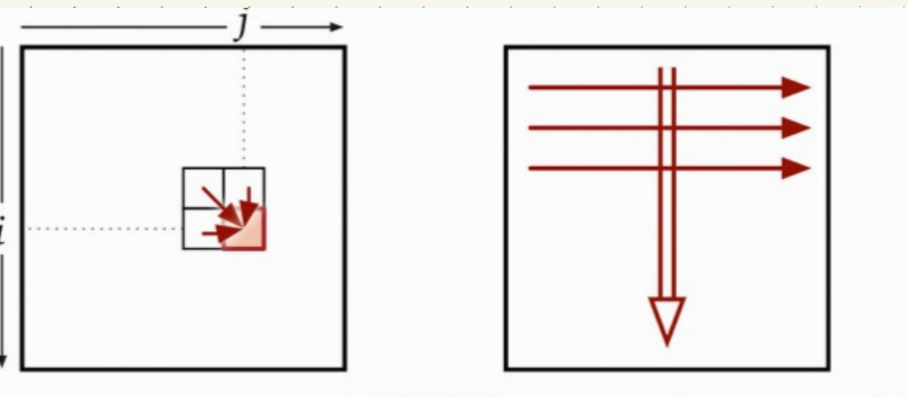
edges:

$(i,j) \rightarrow$

$(i,j) \rightarrow$

Edit distance:
he actually (sort of) showed the graph!

$$
Edit(i,j) = \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \left\{ \begin{array}{c} Edit(i, j-1) + 1 \\ Edit(i-1, j) + 1 \\ Edit(i-1, j-1) + [A[i] \neq B[j]] \end{array} \right\} & \text{otherwise} \end{cases}
$$

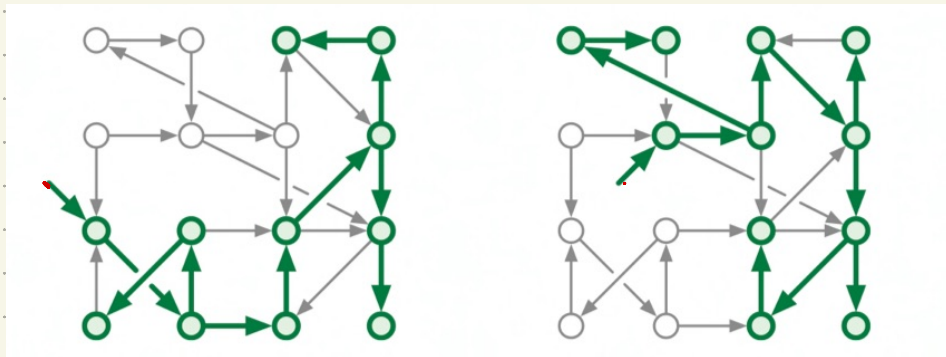


		A	L	G	O	R	I	T	H	M
	0	1	2	3	4	5	6	7	8	9
A	1	**0**	1	2	3	4	5	6	7	8
L	2	1	**0**	1	2	3	4	5	6	7
T	3	2	1	1	2	3	4	**4**	5	6
R	4	3	2	2	2	**2**	3	4	5	6
U	5	4	3	3	3	3	3	4	5	6
I	6	5	4	4	4	4	**3**	4	5	6
S	7	6	5	5	5	5	4	4	5	6
T	8	7	6	6	6	6	5	**4**	5	6
I	9	8	7	7	7	7	**6**	5	5	6
C	10	9	8	8	8	8	7	6	6	6

Strong connectivity

In an undirected graph, if $u \rightsquigarrow v$, then $v \rightsquigarrow u$.

Not true in directed case:



So 2 notions:

weak connectivity:

strong connectivity:

related: SCCs

Can actually order the strongly connected pieces of a graph:

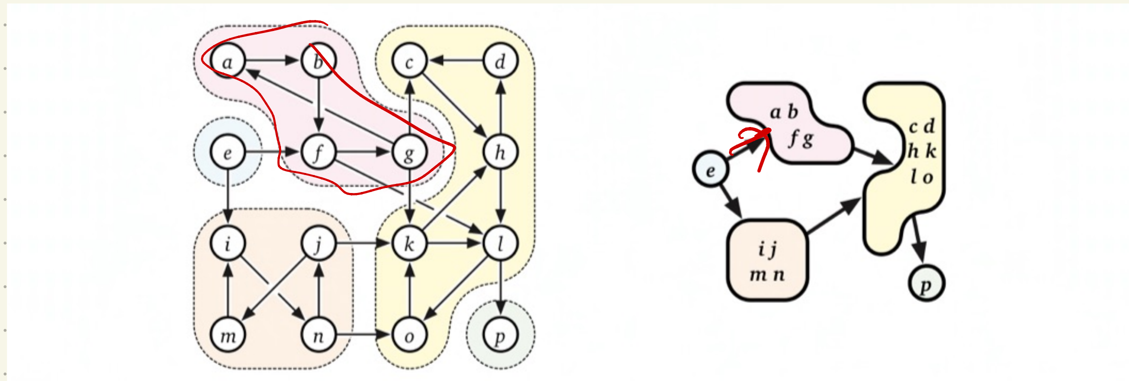


Figure 6.13. The strong components of a graph G and the strong component graph $scc(G)$.

How?

- Well, each component either isn't connected, or only has 1-way edges. Why?

scc scc

Possible to compute SCCs in $O(V+E)$ time.

Sorry - did <u>not</u> assign this one!

But feel free to read anyway. :)

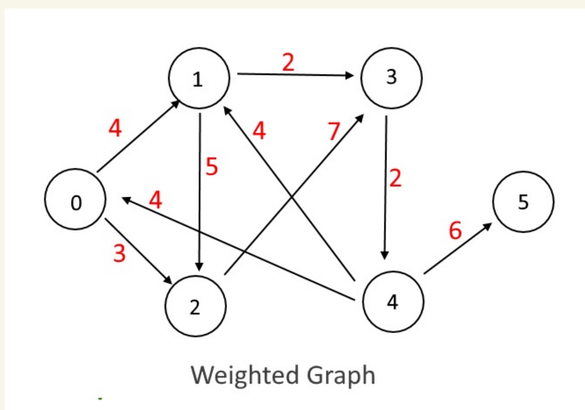
Next module:

Minimum Spanning trees

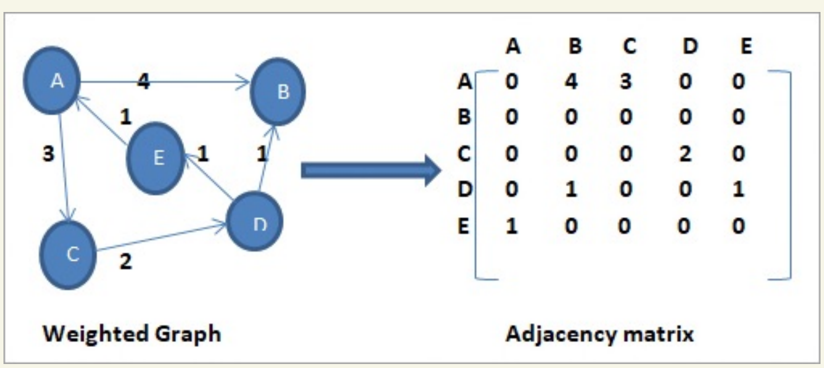
& shortest paths.

Both are on <u>weighted</u> graphs – so $G=(V,E)$, plus $w: E \to \mathbb{R}$ (or $\mathbb{R}^+$)

<u>picture</u>:



Weighted Graph



Weighted Graph

Adjacency matrix