CS443 - More Crypto

Note Title 1/18/2013

Announcements

- Lab 1 due tomorrow
- No class Thursday
  Instead, read "No Tech Hacking"
  & essay due in class Tuesday.
- HW2 - over cryptography.
  due in 2 weeks

Last time: DES & AES

Symmetric key cryptography:

- based on shared knowledge of a private key
- very secure: 128-bit key in AES would take roughly 10 billion billion (est. 1.02×10^{18}) years to crack on a 2012 supercomputer

Assymetric key Cryptography

- based on 1-way communication
- huge development, first published in "New directions in cryptography" by Diffie & Hellman, 1976

Daily conspiracy:
Actually, secretly developed by government researchers in the U.K. in 1973.

General overview

(1) Alice looks up Bob's public key E_B in a directory.
Uses an algorithm to encode message m: written $E_B(m)$

(2) Bob then uses private key D_B to decode: $D_B(E_B(m)) = m$

Idea: Alice can give away m & $E_B(m)$, but no one else can decode without D_B.

Number Theory

2 & 8 are not relatively prime

3 and 11 are

The Euler phi function $\phi(n)$ is defined so that

$\phi(n) = \#$ of integers $\leq n$ that are $\underbrace{\text{relatively prime}}_{\text{no common divisors}}$ to n.

Ex: $\phi(7) = 6$ $\quad (\phi(7) = 7)$

$\phi(6) = \cancel{3}\, 2$

1 and 5

Lemma: If p is prime, $\phi(p) = p-1$

Lemma: If p & q are prime,

$$\phi(p \cdot q) = (p-1)(q-1)$$
$$= \phi(p)\,\phi(q)$$

Why? $p \cdot q$ divisors: $p, 2p, 3p, \ldots (q-1)\cdot p$

$q, 2q, \ldots (p-1)\cdot q$

Euler's thm: If n is a positive integer with $\underbrace{\gcd(a,n)=1}_{\substack{\uparrow \\ a \,\&\, n \text{ are relatively prime}}}$,

then $\underbrace{a^{\phi(n)} = 1}$ mod n

Cor: If a is relatively prime to p & q (both primes), then

$$\underbrace{a^{(p-1)(q-1)}}_{\substack{\| \\ a^{\phi(pq)}}} = 1 \pmod{pq}$$

Remember $\mathbb{Z}_n$?

Key: If a is relatively prime to n, then $\exists b$ with $ab = 1 \mod n$

Ex: $n = 8$

$a = 2$: $2x = 1 \mod 8 \rightarrow$ NO inverse

$a = 3$: $3x = 1 \mod 8 \Rightarrow x = 3$

$a = 4$: No inverse

$a = 5$: $5 \cdot x = 1 \mod \Rightarrow x = 5$

$a = 6$: No inverse

$a = 7$: $7 \cdot x = 1 \mod 8 \Rightarrow x = 7$

RSA: [Rivest - Shamir - Adelmann 1978]

① Bob generates 2 primes p & q and computes $n = p \cdot q$
$\phi(n) = (p-1)(q-1)$

② Bob picks e relatively prime to $\phi(n)$, & then finds d s.t. $ed \equiv 1 \bmod \phi(n)$
(via Euclidean algorithm)

→ d is private key

→ (e, n) are public key

Side note: Euclidean Algorithm

Input: a, b
While $b \neq 0$
    $r \leftarrow a \bmod b$
    $a \leftarrow b$
    $b \leftarrow r$
Return a

What does it do?

calculates gcd of a & b

So: given e & $\phi(n)$ relatively prime,

then $\gcd(e, \phi(n)) = 1$

By tracking variables in this algorithm, get value of d.

Now: Alice has a message m.
To send it to Bob:

1. Compute $c = m^e \bmod n$ — (e, n) is public
2. Send to Bob

Bob decodes:

$$c^d = (m^e)^d \bmod n$$

$$= m^{ed} \bmod n \stackrel{?}{=} m$$

Why it works:

- know $ed \equiv 1 \mod \phi(n)$

  so $ed = \boxed{1 + k\phi(n)}$ for some k

  $= 1 + k(p-1)(q-1)$

- Then $m^{ed} \equiv m^{1+k\phi(n)} \mod n$

  $\equiv (m^1)(m^{k\phi(n)}) \mod n$

  $\equiv m \cdot \left(m^{\phi(n)}\right)^k \mod n$

  $\equiv m \, (1)^k \mod n \equiv m$

Public: (e, n)

How hard to get d?

- Easy if you know $\phi(n) = (p-1)(q-1)$

need $ed \equiv 1 \bmod \phi(n)$

Need to factor n to find p & q

Notes

- Actually, in general no one will know p or q - use central certificate authority.

Public: (e, n)
Private: d

How do we get d again?

↳ Simple with p & q.
"Hard" without.

This is why the effectiveness of RSA is based on factoring!

If we knew $\phi(n) = (p-1)(q-1)$, could break the system.

How hard is factoring?

No 512-bit number has (yet) been factored.

Diffie-Hellman key exchange [1976]

Most common use of public key cryptography is to exchange private keys!

Why?

Symmetric encryption is more secure.

Diffie-Hellman basics:

Consider a prime number q (or $q = p^k$, with p prime).

We saw last time that $\mathbb{Z} \bmod q$ is a finite field:

- nice + & × operation
- has <u>multiplicative</u> <u>inverse</u>:

<u>Ex</u>: $2 \cdot x = 1 \mod 5$

$$\Rightarrow x = 3$$

The protocol: p prime, $s < p$ (both public)

- Alice chooses $a < p$
  Bob chooses $b < p$

- Alice computes $\alpha = s^a \bmod p$
  Bob computes $\beta = s^b \bmod p$

- They exchange α and β

- Alice computes $\beta^a \bmod p$
  Bob computes $\alpha^b \bmod p$

$$\beta^a = (s^b)^a \bmod p = s^{ab} \bmod p$$
$$\alpha^b = (s^a)^b \bmod p = s^{ab} \bmod p$$

secret key

Ex: Let $s = 2$, $p = 29$

Alice likes $a = 14$

Bob likes $b = 12$

$$\alpha = 2^{14} \bmod 29 = 28$$

$$\beta = 2^{12} \bmod 29 = 7$$

$$\alpha^{12} \bmod 29 = 28^{12} \bmod 29 = 1$$

$$\beta^{14} \bmod 29 = 7^{14} \bmod 29 = 1$$

Common key $k = s^{ab} \bmod p$

Recap: Public info: • p and s

• $\alpha = s^a \bmod p$

• $\beta = s^b \bmod p$

Private: a (to Alice)

b (to Bob)

k

$$\alpha \cdot \beta = (s^a)(s^b) \bmod p = \underbrace{s^{a+b}}_{\neq k} \bmod p$$

Why is it hard to break?

At its base, the key is logarithms.
(Remember those?)

$\log_{10} 1000 = 3$

$\log_2 1024 = 10$

We want discrete logs:
given α, find a.

$\log_s(s^a) \bmod p$

The Discrete Log Problem

This is another one we "think" is hard.

Similar to factoring.

Note: There are ways to attack this! Not NP-Hard - just no known fast algorithms.

Stonger generalizations work in groups other than $\mathbb{Z}_p$.

(But elliptic curves are a bit beyond us now...)

Bigger Picture : NSA Suite B

The NSA has published a set of recommended algorithms (for both unclassified information as well as info up to SECRET).

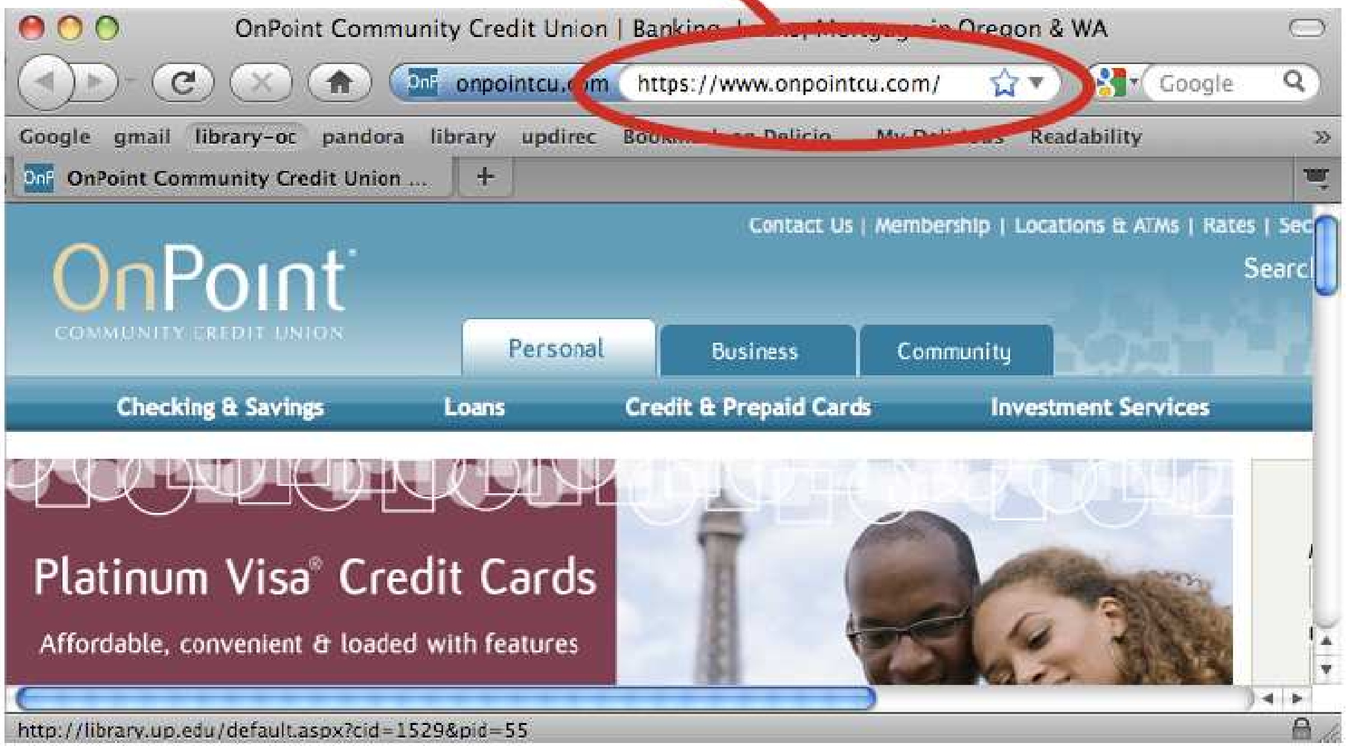
- Encryption : AES
- Signatures : Elliptic curve Diffie-Hellman
- Hashing : SHA (Secure hashing algorithm)

So- why study RSA??

Whenever you see "https", that's TLS at work.

Still in use!

(RSA is the basis of the Transport Layer Security in browsers.)



Why RSA: Cost

(Also used in smart cards, operating systems, etc.)

But why, if ECDH is better??

Most companies can't afford it, since the patents are still largely held by one company.

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