# CS 314 - Lecture 3

Note Title 1/15/2010

## Announcements

- HW0 due Wednesday
- Pdfs of lectures should be up after class today
- Wednesday we start actual algorithms! (no class Monday)

<u>Today</u> – Recursion

If induction starts at base case & builds up, then recursion is <u>dual</u> idea:

IS – Start with $n$ things

IH – Reduce to (a) smaller subproblem(s)

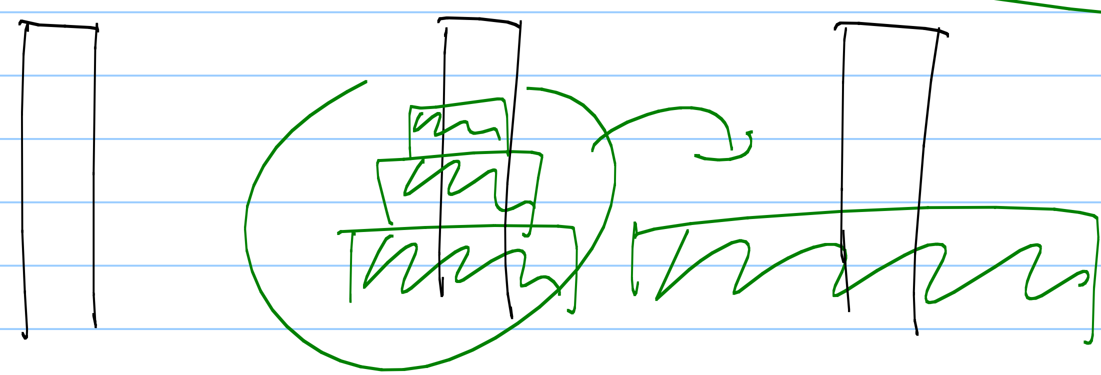
BC – Eventually reach smallest case
<u>aka</u> base case.

# Solving recurrences

**Method 1: guess & check**

Ex: Towers of Hanoi

→ $T(n) = T(n-1) + 1 + T(n-1)$



→ $2^n - 1$ moves

How many moves? $2^n - 1$

Write as recurrence:

$$T(n) = 2T(n-1) + 1$$

Fill in some values:

$T(1) = 1$ $\quad T(2) = 3$ $\quad T(3) = 2(3)+1 = 7$

$T(4) = 2(7)+1 = 15$ $\quad T(5) = 2(15)+1 = 31$ $\quad T(6) = 31(2)+1 = 63$

...

$T(n) = 2^n - 1$

Proof: (by induction)

BC: $n=1$   $T(1)=1$ (only 1 disk!) ✓

$2^1 - 1 = 1$

IH: For $\underset{n-1<n}{k<n}$ disks, $T(k) = 2^k - 1$.

$\xrightarrow{k=n-1}$ $T(n-1) = 2^{n-1} - 1$

IS: Consider $n$ disks.

Use recurrence:

$$T(n) = 2T(n-1) + 1$$

apply IH $= 2(2^{n-1} - 1) + 1 = 2^n - 2 + 1$

$= 2^n - 1$ $\blacksquare$

Guess + check - mergesort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$

Answer?

$$T(n) = \underline{O}(n \log n)$$

$$\underline{\Theta} \longleftrightarrow \underline{O, \Omega}$$

Need to strip off big-O to use induction
(but ±1 will not change big-O, unless in exponent)

Assume $n =$ power of 2

$T(n) = 2T(n/2) + n$, $T(1) = 1$

$T(k) = 2T\left(\frac{k}{2}\right) + k$

$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{n}{2}$

unroll:

$$T(n) = \overset{2^1}{2}\,T\left(\frac{n}{2}\right) + n$$
$$= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$
$$= \overset{2^2}{4}\,T\left(\frac{n}{4}\right) + 2n$$
$$= 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$
$$= \overset{2^3}{8}\,T\left(\frac{n}{8}\right) + 3n$$
$$= 8\left(2T\left(\frac{n}{16}\right) + \frac{n}{8}\right) + 3n$$

$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{n}{4}$

$T\left(\frac{n}{8}\right) = 2T\left(\frac{n}{16}\right) + \frac{n}{8}$

$$= \underbrace{16}_{2^4}\, T\left(\frac{n}{16}\right) + 4n$$

$$\vdots$$

$$= 2^i\, T\left(\frac{n}{2^i}\right) + in$$

Base case: $T(1) = 1$

happens when: $\frac{n}{2^d} = 1 \implies n = 2^d$

$\lg n = \lg 2^d = d \cdot \lg 2 = d$

(then $T(1) = T\left(\frac{n}{2^d}\right)$)

$\boxed{d = \lg n}$

$$T(n) = 2^d\, T\left(\frac{n}{2^d}\right) + d \cdot n = 2^{\lg n} \cdot 1 + (\lg n) \cdot n = n + n \lg n$$

# Recursion Trees

Divide + conquer recurrences often look like:

$$T(n) = aT(n/b) + f(n)$$

$a, b$ constant, $f(n)$ a function

unroll:

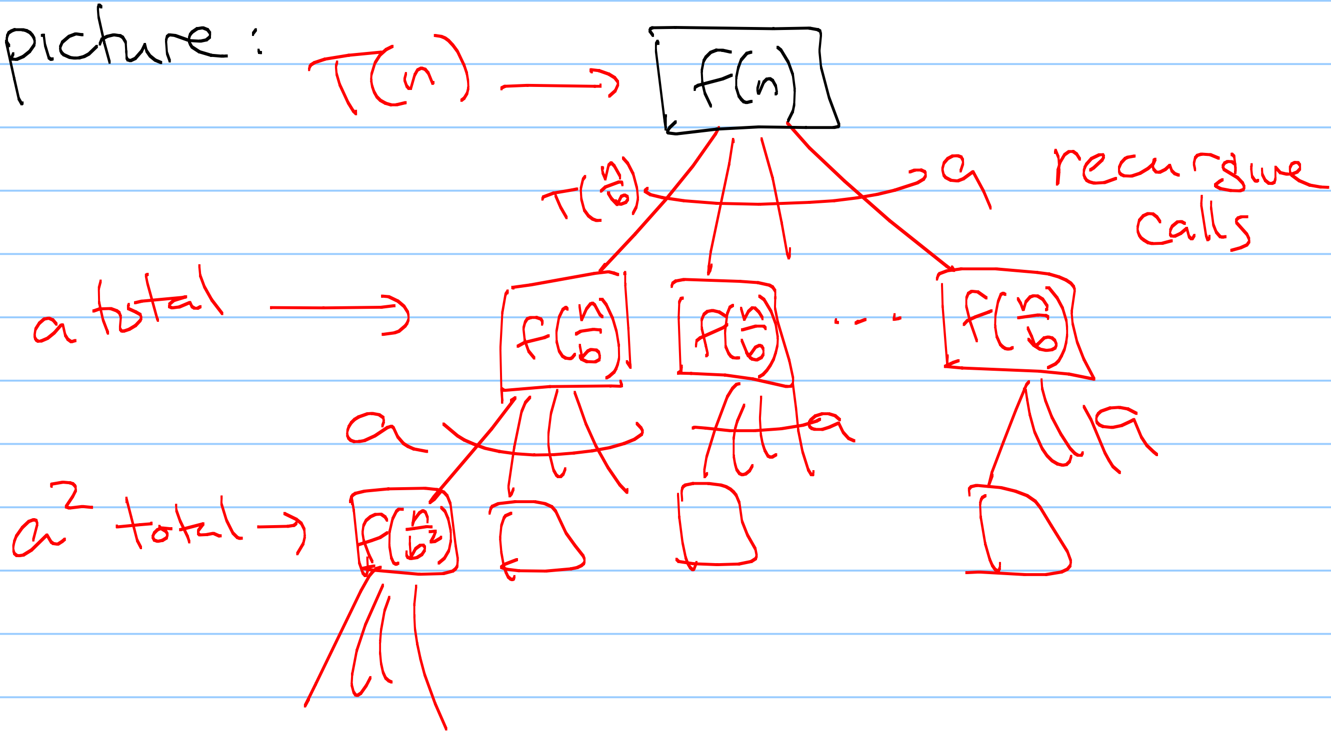
$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$= a\left(aT\left(\frac{n}{b^2}\right) + f\left(\frac{n}{b}\right)\right) + f(n)$$

$$\vdots$$

$$T(k) = aT\left(\frac{k}{b}\right) + f(k)$$

picture:



Gives a sum! (& we like those...)

$a = b = 2$

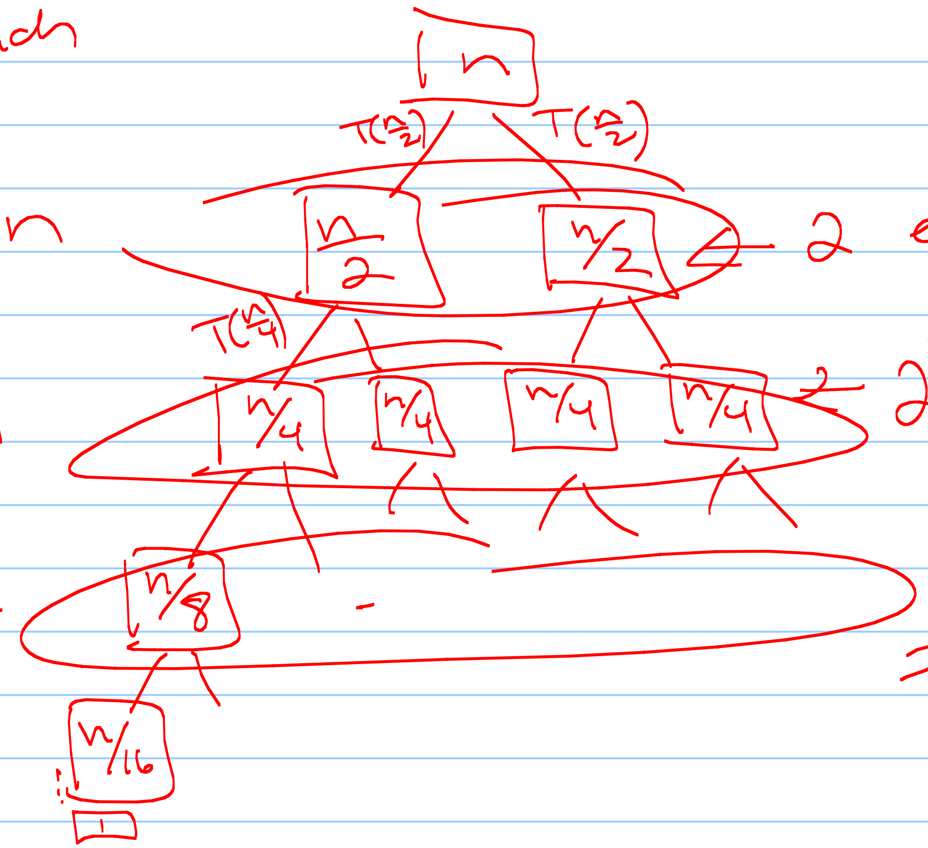
**Ex:** Mergesort $T(n) = 2T(\frac{n}{2}) + n$

# elts / work in each

$$\sum_{i=1}^{d} 2^i \left(\frac{n}{2^i}\right)$$

$$= \sum_{i=1}^{d} n$$

$$= d \cdot n$$




← 2 elements of size $n/2$

← $2^2$ elements of size $\frac{n}{2^2}$

$\frac{n}{2^d} = 1$

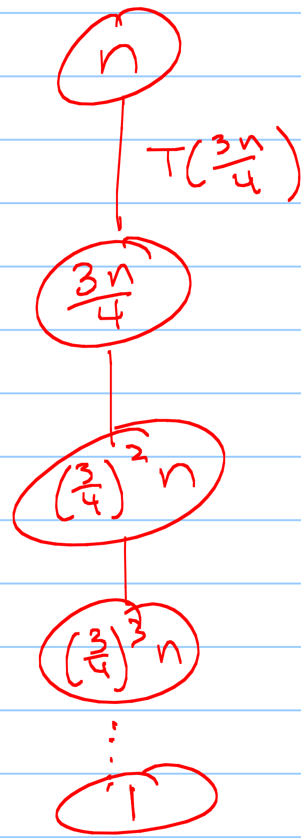
$\Rightarrow d = \lg n$

Randomized selection: $T(n) = T(3n/4) + n$

$a = 1$

$$\sum_{i=1}^{d} 1 \cdot \left(\frac{3}{4}\right)^{i} \cdot n$$

(geometric sum!)




depth d

$$\left(\frac{3}{4}\right)^{d} n = 1$$

$$n = \left(\frac{4}{3}\right)^{d}$$

$$\log_{4/3} n = \log_{4/3} \left(\frac{4}{3}\right)^{d} = d$$

$$\boxed{d = \Theta(\log_2 n)}$$

Master Thm: Spps $T(n) = aT(n/b) + f(n)$

1) If $af(\frac{n}{b}) = k \cdot f(n)$ for $k < 1$, then
$$T(n) = \Theta(f(n))$$

2) If $af(\frac{n}{b}) = k \cdot f(n)$ for $k > 1$, then
$$T(n) = \Theta(n^{\log_b a})$$

3) If $af(n/b) = f(n)$, then
$$T(n) = \Theta(f(n) \cdot \log_b n)$$

Why?

geometric series!!

Caution: Not all recurrences satisfy Master Thm!

Can use recursion trees even for those.

Ex: $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$