CS314: Huffman codes

Note Title 2/12/2010

Announcements

- Next week is oral grading - sign up Monday
- HW is out
- Midterm 1 - Friday ~~the 5th~~
  (right before break)
  or Friday the 19th
  (right after break)

Binary encodings

How do computers encode characters?

binary!

ASCII - each character gets 8 bits

$01000001 = (41)_{16}$ - A

$(42)_{16}$ - B

What if sending a bit is expensive?

Won't use same # of bits per letter, since some (s, t, e, a) are much more common than others (z, q).

Leads to <u>data</u> <u>compression</u>.

Oldest form (at least common in US):
Morse code

e $\overset{0}{\bullet}$, t $\overset{1}{-}$, a $\overset{0}{\bullet}\overset{1}{-}$

(really trinary since uses pause)

Disadvantage: How to translate 0101 ?

$\bullet - \bullet -$

a a
e t
e t a

ambiguous

Ambiguity comes from one string being a prefix of another.

→ (So first 0 could be an e, or could start an a)

Prefix-free codes: No letter's code is a prefix of another.

Advantage: - as we scan, each possibility is unique
- quick-linear scan

Example: MISSISSIPPI

M : 111
→ I : 10
→ S : 0
P : 110

S : 4 ←
I : 4 ←
→ P : 2
M : 1

~~S : 0~~
~~I : 1~~
~~done – no more letters~~

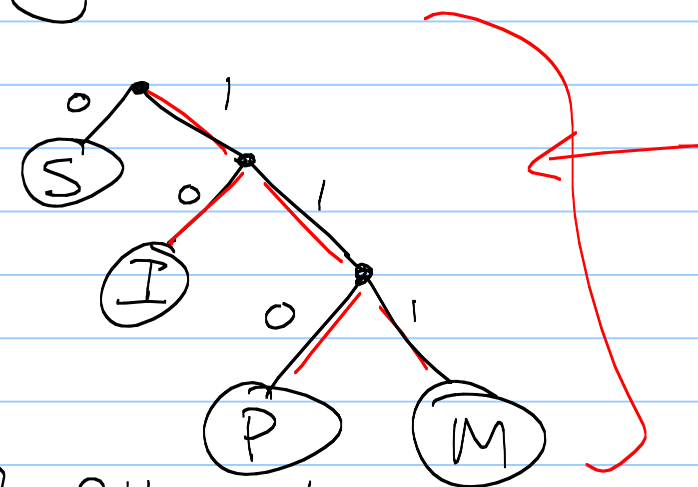
Encoding: $\underbrace{111}_{M}\underbrace{10}_{I}\underbrace{0}_{S}\underbrace{0}_{S}$10001001011011010

21 bits

(ASCII: 88 bits)

Can visualize as a binary tree

M : 111
I : 10
S : 0
P : 110



When reading string, just follow tree + output letter when you reach a leaf.

<u>Ex</u> : 0 10 111 110

S I M P

Question: How can we find minimal prefix free codes?

Given n letters plus frequency counts for each letter, $f[1..n]$, find the code minimizing total length:

$$\underbrace{\sum_{i=1}^{n} f[i] \cdot \text{depth}(i)} = \text{cost}(T)$$

Huffman codes (1952)

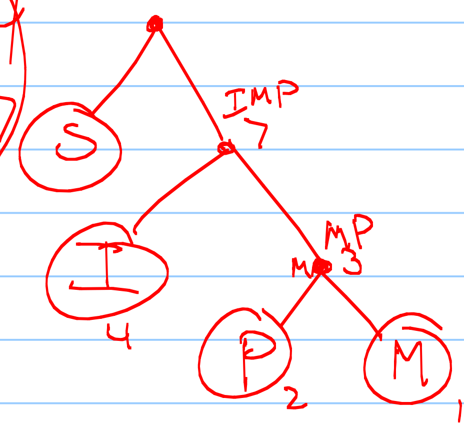
Huffman designed a greedy algorithm:
Merge the two least frequent letters and recurse.

S : 4
I : 4
P : 2
M : 1

→

S : 4
I : 4
PM : 3

S : 4
IMP : 7



→ This is optimal!

Pseudocode:

- Keep characters in min heap, w/priority = to frequency.
- L, R, & P keep track of left/right and parent indices.

```
BUILDHUFFMAN(f[1..n]):
    for i ← 1 to n
        L[i] ← 0; R[i] ← 0
        INSERT(i, f[i])

    for i ← n to 2n − 1
        x ← EXTRACTMIN()
        y ← EXTRACTMIN()
        f[i] ← f[x] + f[y]
        L[i] ← x; R[i] ← y
        P[x] ← i; P[y] ← i
        INSERT(i, f[i])

    P[2n − 1] ← 0
```

Example:

This sentence contains three a's, three c's, two d's, twenty-six e's, five f's, three g's, eight h's, thirteen i's, two l's, sixteen n's, nine o's, six r's, twenty-seven s's, twenty-two t's, two u's, five v's, eight w's, four x's, five y's, and only one z.

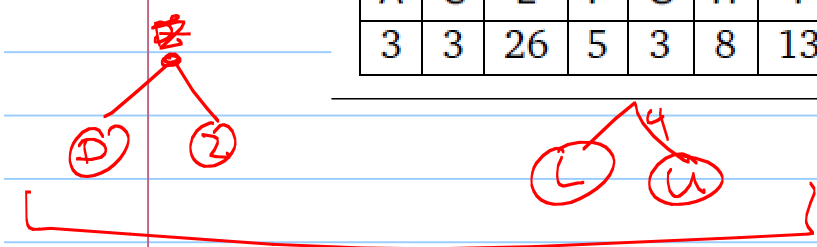
Frequency counts:

A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1

Two least frequent? combine D + Z, + they become leaves
frequency of DZ = 3

So take this & merge:

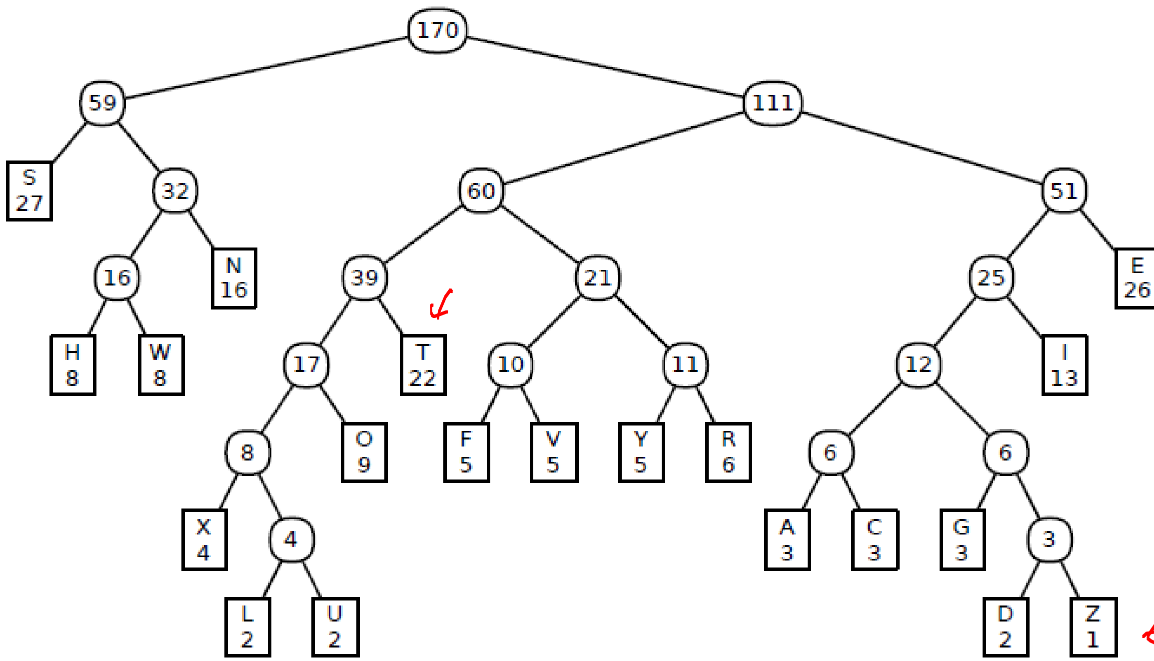
A	C	E	F	G	H	I	~~L~~	N	O	R	S	T	~~U~~	V	W	X	Y	DZ	LU
3	3	26	5	3	8	13	~~2~~	16	9	6	27	22	~~2~~	5	8	4	5	3	4



Next?

etc...

This gives us the tree:



Then L + U

Note - D + Z were first 2 merged

Encoding:

1001	0100	1101	00	00	111	011	1001	111	011	110001	111	110001	10001	011	1001	110000	1101	…
T	H	I	S	S	E	N	T	E	N	C	E	C	O	N	T	A	I	

Total: 646 bits (versus 1,450 for ASCII)

char.	A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
freq.	3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1
depth	6	6	7	3	5	6	4	4	7	3	4	4	2	4	7	5	4	6	5	7
total	18	18	14	78	25	18	32	52	14	48	36	24	54	88	14	25	32	24	25	7

<u>Claim</u>: This is optimal!

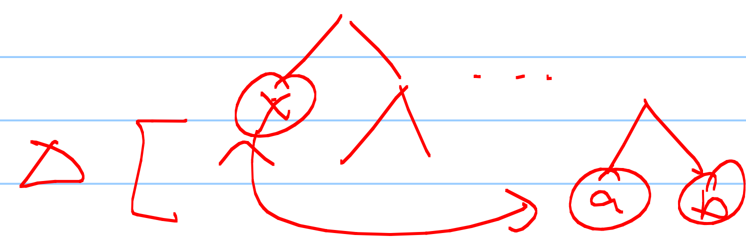
(<u>No other</u> encoding could do better.)

Proof of Correctness:

Lemma: Let x & y be the 2 least frequent characters. Then there is an optimal code where x & y are siblings and have maximum depth in the tree.

proof: Let T be an optimal tree, with depth d. T must have at least 2 siblings at depth d. If x & y are there, done. So assume a & b are these siblings. (not x & y)

Swap a & x.



Depth of $\overbrace{x \text{ increases by some } \Delta \geq 0, \text{ and depth of } a \text{ decreases by } \Delta}$.

$$\text{cost}(T') = \text{cost}(T) \underbrace{- f[a]\cdot\Delta} \underbrace{+ f[x]\cdot\Delta}$$

but $f[a] \geq f[x]$

so $\Delta(f[x] - f[a])$ is positive,

& T' must use ~~fewer~~ no more bits

Similarly, can swap b & y. $\boxed{\text{M}}$

Thm: Huffman codes are optimal.