CS 314 - More graph NP-Hardness

Note Title 3/29/2010

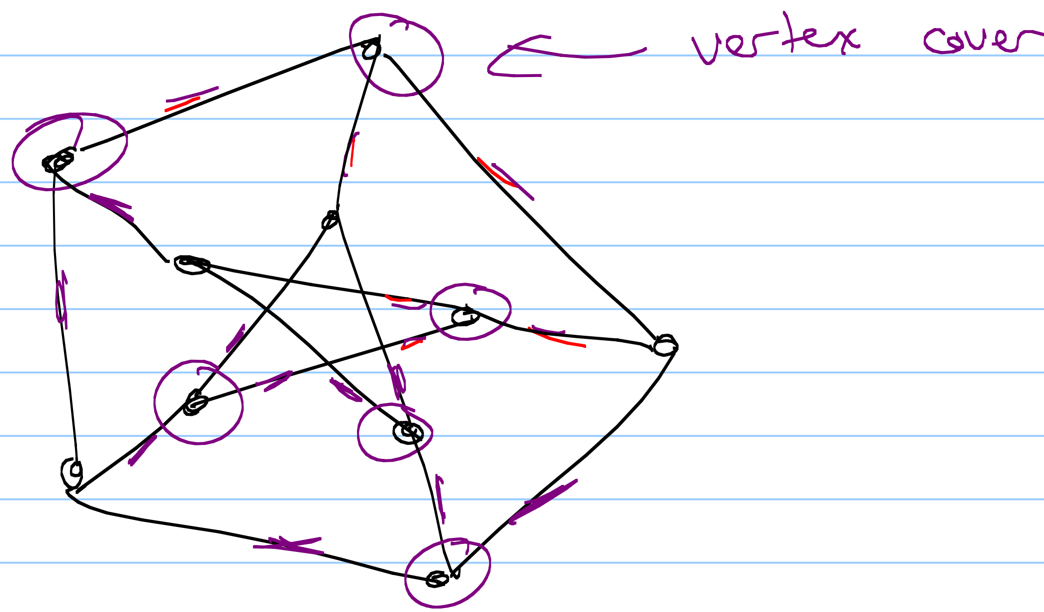
Announcements

- HW out, due (oral grading) next Tuesday
- Next HW will be out Mon./Tues. next week, & will be written NP-Completeness problems.

Vertex Cover

Dfn: A vertex cover is a subset of vertices $S \subseteq V$ such that every edge is adjacent to a vertex in S.

Ex:



Q: Given a graph G and value k, is there a vertex cover of size $\leq k$?

note: $\leq$!

Vertex Cover is NP-Complete:

① VC is in NP:

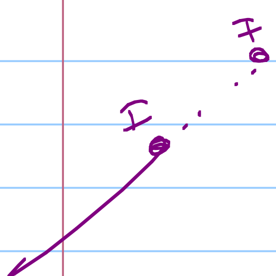
Given k vertices, label edges 1..m.
Mark edges in G that are adjacent to one of the k vertices.
If all edges are marked, the k vertices are a vertex cover.
$O(m+k)$

NP-Hard: C-SAT, 3SAT, SAT, Ind Set, Clique

② V.C. is NP-Hard: $\underbrace{\text{I.S.} \le_p \text{V.C.}}$

Key Fact: If I is an ind. set, then $V - I$ is a vertex cover.

pf: Spps I is an ind. set in a graph G.
So no edges b/t any 2 vertices in I.



$\Rightarrow$ every edge has an endpoint not in I.

So $V - I$ is a vertex cover. ∎

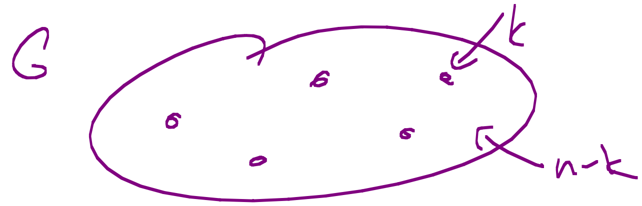
Ind. Set $\le_p$ Vert. Cover:

Input: G, k Want to say yes if G has ind. Set of size $\ge k$.

Transform to another G' & ask is G' has vert. cover of size $\le k'$.

G to have IS of size $\ge k$

$\iff$ G' has V.C. of size $\le k'$



So we don't need to transform G at all!

Given G & k as input to ind. set problem, just ask black box for vertex cover if there is a vertex cover of size n-k.

If ind set of size $\geq k$, then v.c. of size $\leq n-k$.

If v.c. of size $\leq n-k$, then ind set of size $\geq k$.

So here, set $G' = G$
$k' = n-k$

Recap:

Vertex Cover in NP-Complete.

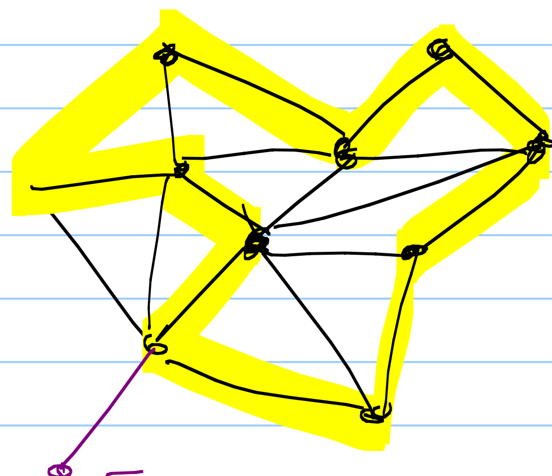
set $k' = n-k$

$$\begin{tikzcd}
\boxed{\text{graph } G=(V,E)} \arrow[r, "\text{trivial}"] & \boxed{\text{graph } G=(V,E)} \arrow[d, Rightarrow, "\textsc{MinVertexCover}"] \\
\boxed{\text{largest independent set } V\setminus S} & \boxed{\text{smallest vertex cover } S} \arrow[l, "O(n)"']
\end{tikzcd}$$

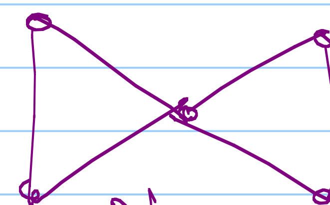
Hamiltonian Cycle

A Hamiltonian cycle is a cycle in a graph which visits every vertex once.

Ex:



Has no Ham. cycle.



No Ham. cycle

Clique SAT
I.S. 3SAT
V.C. C-SAT

Hamiltonian cycle is NP-Complete

① Ham. cycle is in NP

Given a cycle, check visits every vertex once
O(n)

② Reduce 3SAT to Ham. cycle.

(In lec. notes, he reduces Vertex cover to Ham. cycle.)

from Ch. 8

Setup:

We'll show 3SAT $\leq_p$ Ham. cycle.

So we have a "black box" which, given a graph G, will output yes if G contains a Hamiltonian cycle.

Our input is a 3SAT problem.

$\rightarrow (x_1 \vee \overline{x_2} \vee x_{12}) \wedge \underbrace{(x_2 \vee x_5 \vee \overline{x_{26}})} \wedge \ldots$

We need to (in polynomial time) translate that formula into a graph.

G will have 2^n different Ham. cycles possible
(1 for each truth assignment)

Variables $x_1 \ldots x_n$
Clauses $C_1 \ldots C_k$

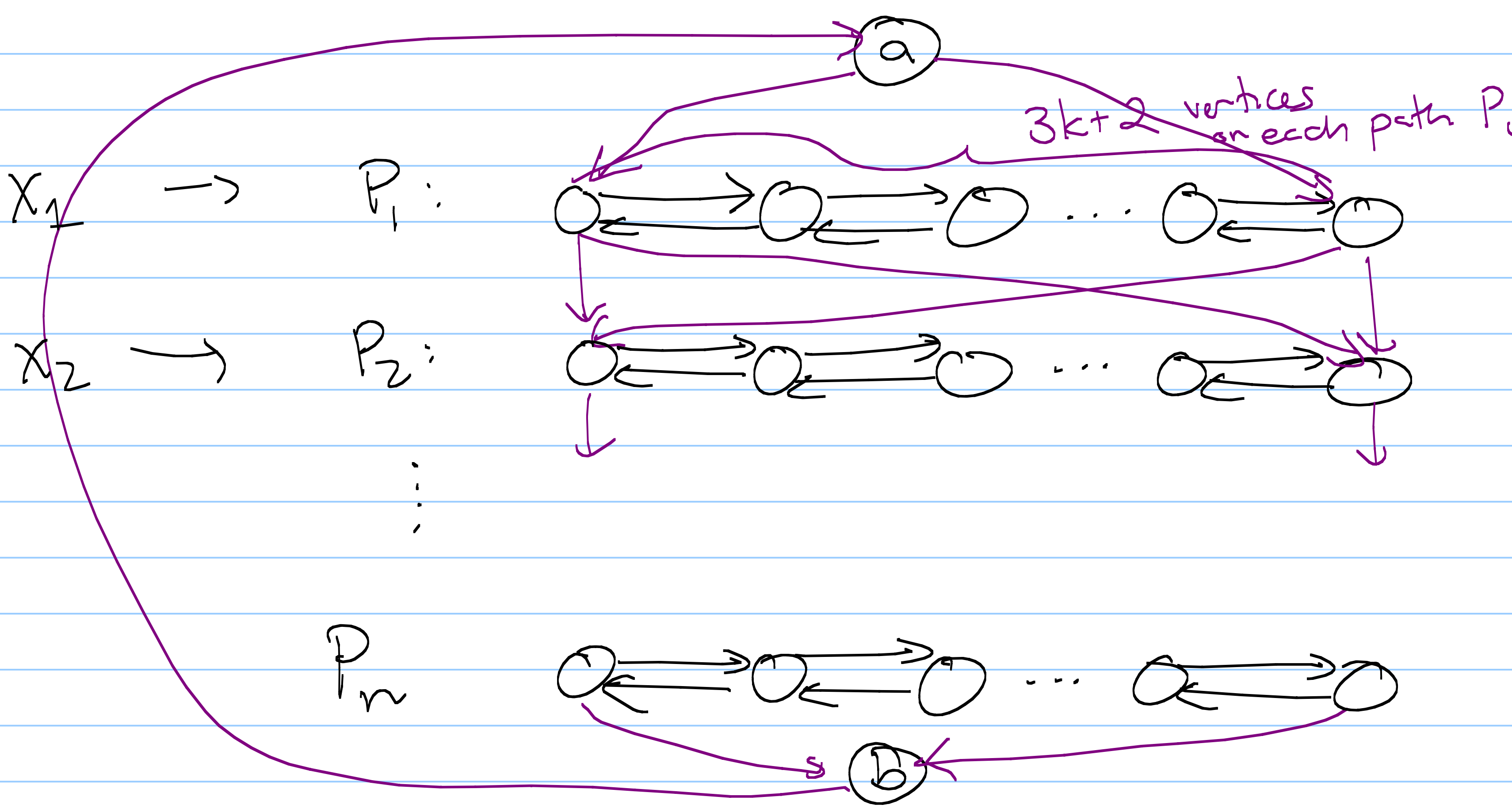
$(\ C_1\ ) \wedge (\ C_2\ ) \wedge (\ C_3\ ) \wedge \ldots (\ C_k\ )$

Each variable will become a path.
→ variable gadges

Each clause will be a vertex.

Variable gadgets

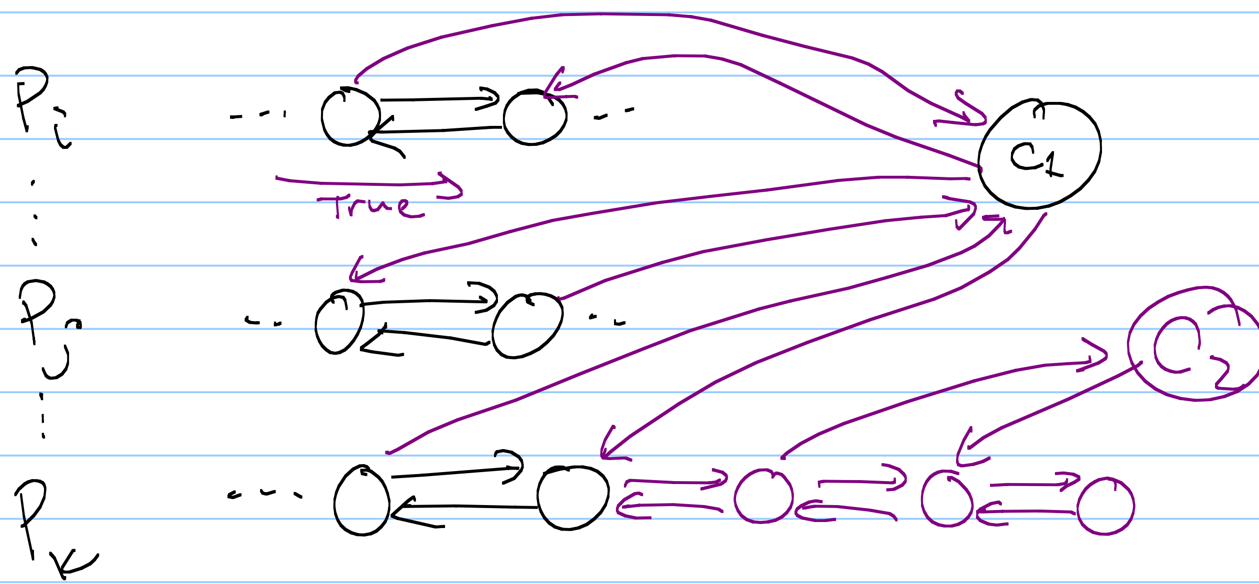
if I go left to right on P_i, then x_i will be "true"



Clause ~~gadgets~~ vertex

Each clause gets 1 vertex & we hook it to the 3 relevant paths:

Clause 1: $x_i \wedge \overline{x_j} \wedge x_k$



(if x_j is false, then $\overline{x_j}$ is true)

(C_2 has x_k in it)

Claim: 3SAT instance is satisfiable
$\iff \exists$ Ham. cycle in G

pf: $\Rightarrow$: Spps there is assignment of $x_1, \dots x_n$ that evaluates to true.
In G, start at a & traverse each P_i left to right if x_i is true, & right to left if x_i is false.

Since formula evaluates to true, at least one variable in each clause is true.
In Ham cycle, I'll visit vertex c_i in the path which corresponds to the true variable.

$\Leftarrow$: Spps Ham cycle C in G.

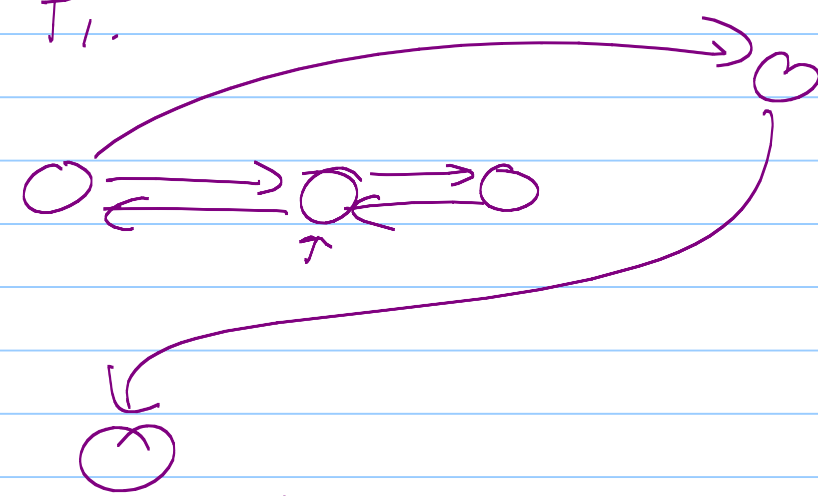
Must use $b \to a$ edge.

Then must go to P_1.

Then P_2

Then P_3

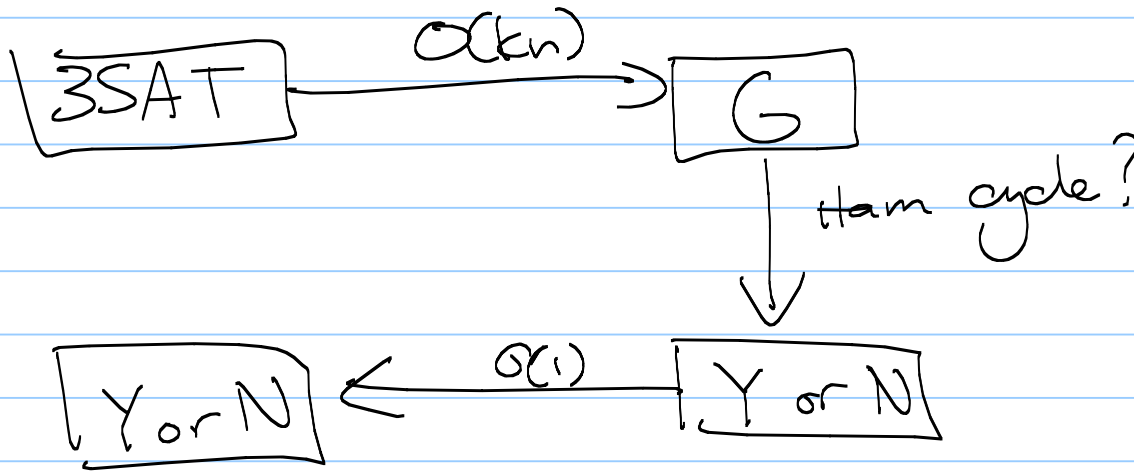
⋮



(full details in Ch. 8)

How long did transformation take?

$O(nk)$


3SAT
O(kn)
G
Ham cycle?
Y or N
O(1)
Y or N

Travelling Salesman Problem

Suppose we have n cities connected by a road network. (complete, directed graph)

Want a tour which visits every city and is as short as possible.

Q: Given a graph G and value k, is there a tour with cost $\leq k$?

TSP is NP-Complete