CS 314 – TSP Approximation

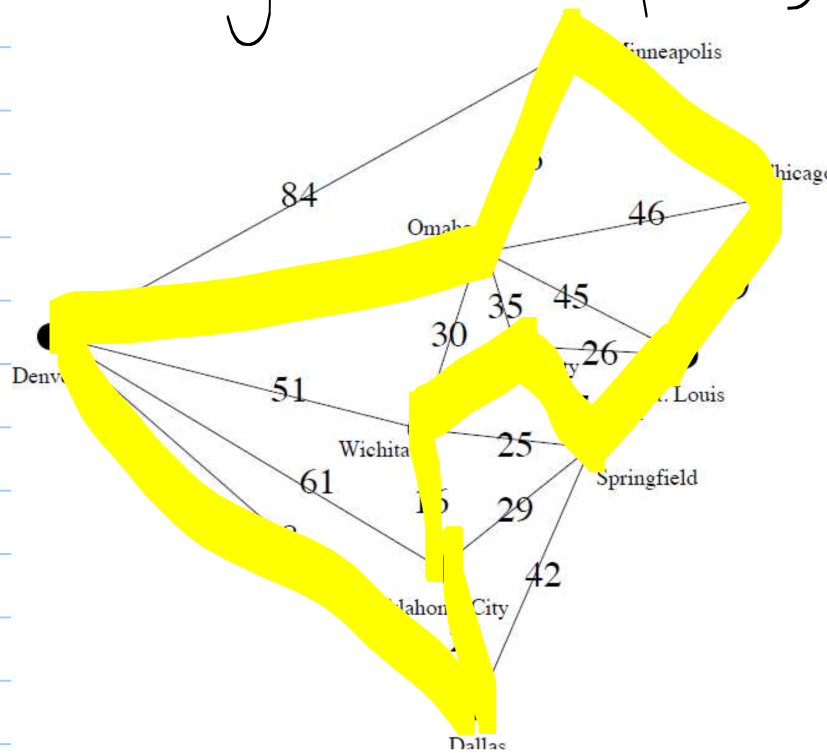
Note Title 4/12/2010

Announcements

- HW due Wed. (written)
- Look for next HW tomorrow
  - oral grading the next Tuesday (hopefully), with backup plan that you submit the written version Wednesday

Traveling Salesman

Q: Is there a Hamiltonian cycle in a weighted (complete) graph with length ≤ k?



← Tour of weight
41 + 36 + 54 + 78 + . . .

TSP is NP-Hard:

Reduction from Ham. cycle.

Input graph G, need to answer yes/no about Ham. cycle.

Create a weighted graph G' on same vertices as G.

- if $e \in G$, then set $w(e) = 1$
- if $e \notin G$, then set $w(e) = 2$.

G has a Ham. cycle $\iff$ G' has TSP of weight $\le n$. ; K

Bad News

f(n)-approx for any polynomially computable function f

Even approximating TSP is NP-Hard!

Consider reduction, but set "absent" edges to weight $n+1$.

Say we had a 2-approx,

A Ham. cycle in G will still be a TSP of weight $= n$.

Any other tour has weight $> 2n$.

A 2-approx in this graph means I'll never use an $n+1$ weight edge.

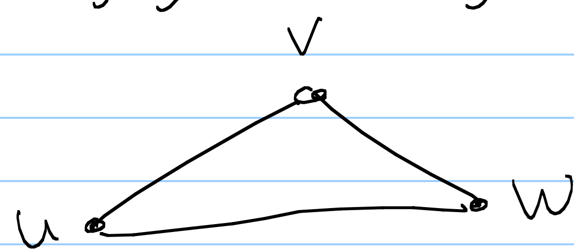
Good News

In some restricted cases, we can approximate a TSP!

We need edges to satisfy the triangle inequality:

for any vertices $u, v, \& w$,

$$e(u,w) \le e(u,v) + e(v,w)$$



When does triangle inequality hold?

- in the plane (Euclidean space)
- geometric graphs

(Note that it doesn't in our reduction, which makes sense.)

Idea:

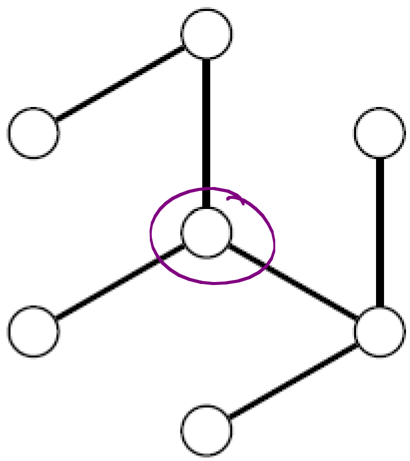
Use a tool we already have!

What can we compute that connects all the vertices + uses a weight that is as small as possible?

MST - connects all vertices
- small as possible

OK, so take a MST:

MST

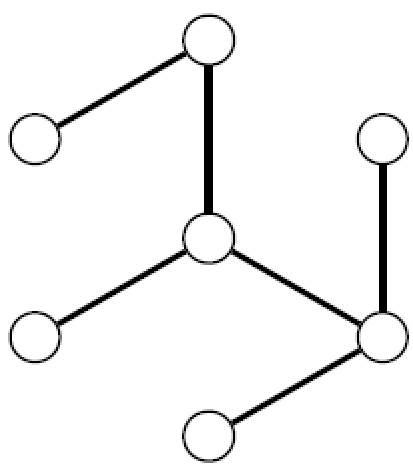


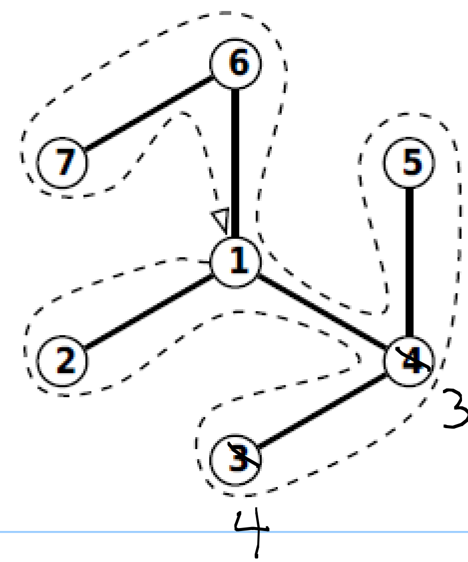
How can we make this a cycle?

Traverse this tree, but convert into cycle.

Label an order to visit the nodes – say DFS, so we "backtrack" as little as possible.

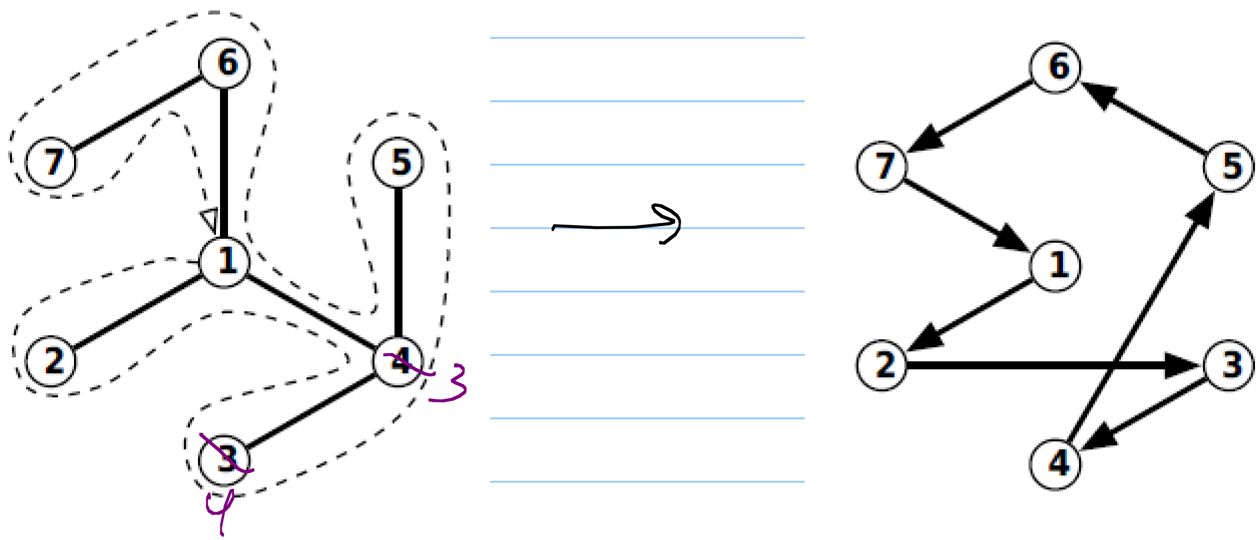
weight MST →





How long is this? $2 \cdot MST$

Now just visit the nodes in that order.



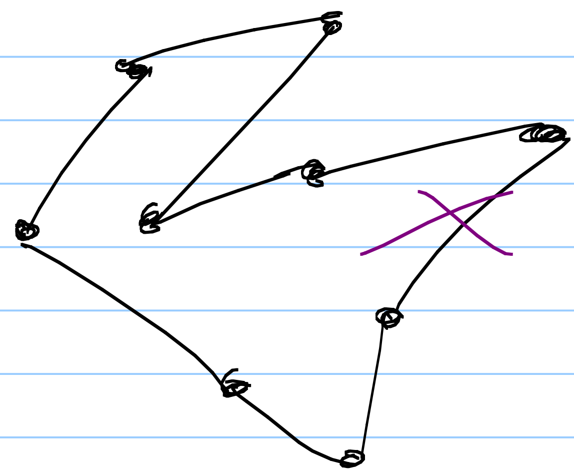
← cycle with weight $\leq 2 \cdot MST$

How much longer did it get?

triangle inequality $\Rightarrow$ $\leq 2 \cdot MST$

One final inequality:

How does the weight of an MST relate to OPT, the weight of a TSP tour?



$$\boxed{MST \le OPT}$$

since deleting an edge from TSP tour gives a tree.

Conclusion:

Our alg returns a tour of weight $A(x) \le \underbrace{2 \cdot MST(x)}$ ①

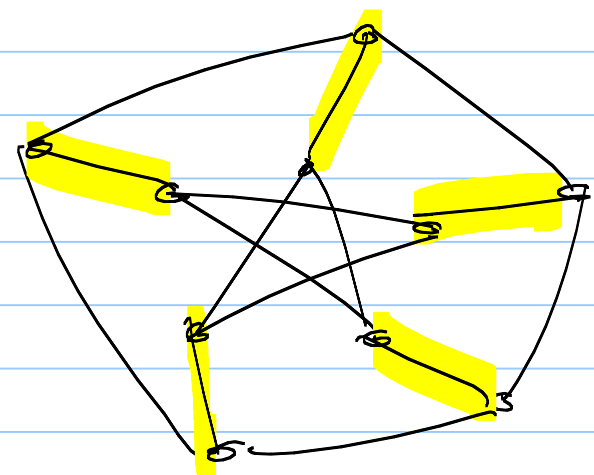
And $MST(x) \le TSP(x)$ ②

$\Rightarrow A(x) \le \underbrace{2 \cdot TSP(x)}$

so we have a 2-approxamation!

A better one!

Dfn: A perfect matching in a graph is a collection of edges where each vertex is adjacent to exactly one edge.

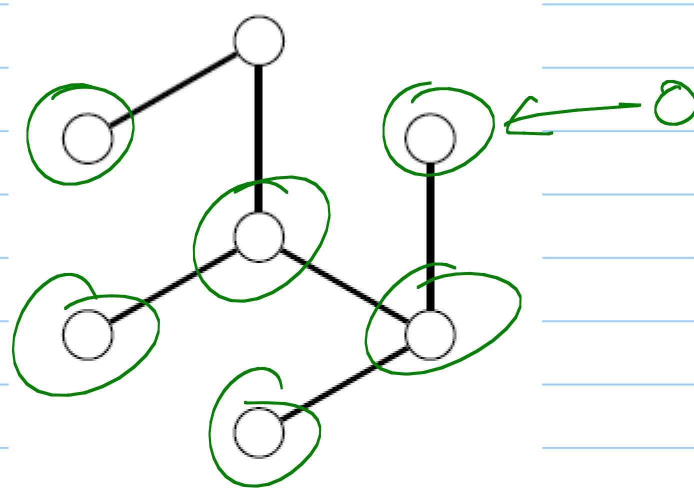


we will talk about computing these using maximum flow, our next section

Let O = set of vertices of MST T with odd degree.

(Note: $|O|$ must be even. Why?)

MST →



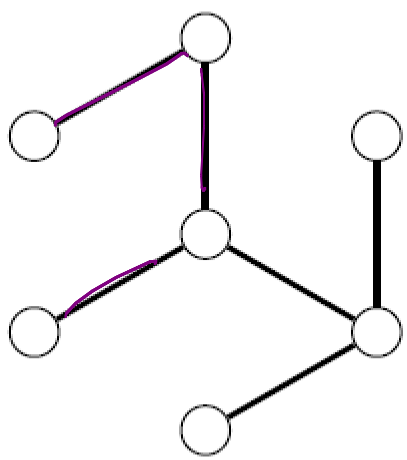
degree-sum

$$\underbrace{\sum d(v)}_{\text{degree-sum}} = \text{even \#}$$

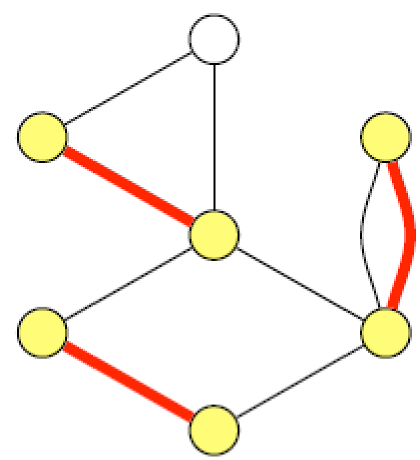
Compute a minimum cost matching M in G for O.

Let T = min. spanning tree.

Consider $T \cup M$ (a multigraph).



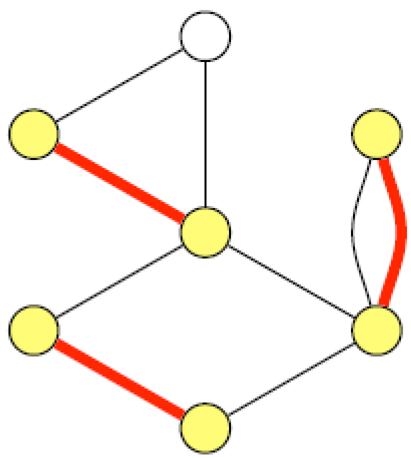
$T \cup M$

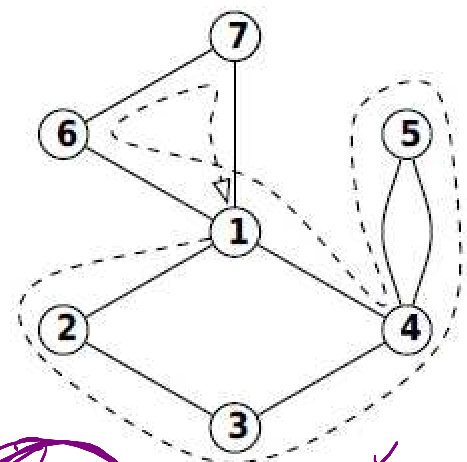


$T \cup O$ is connected, & every vertex has even degree

$\Rightarrow$ contains an Eulerian circuit.

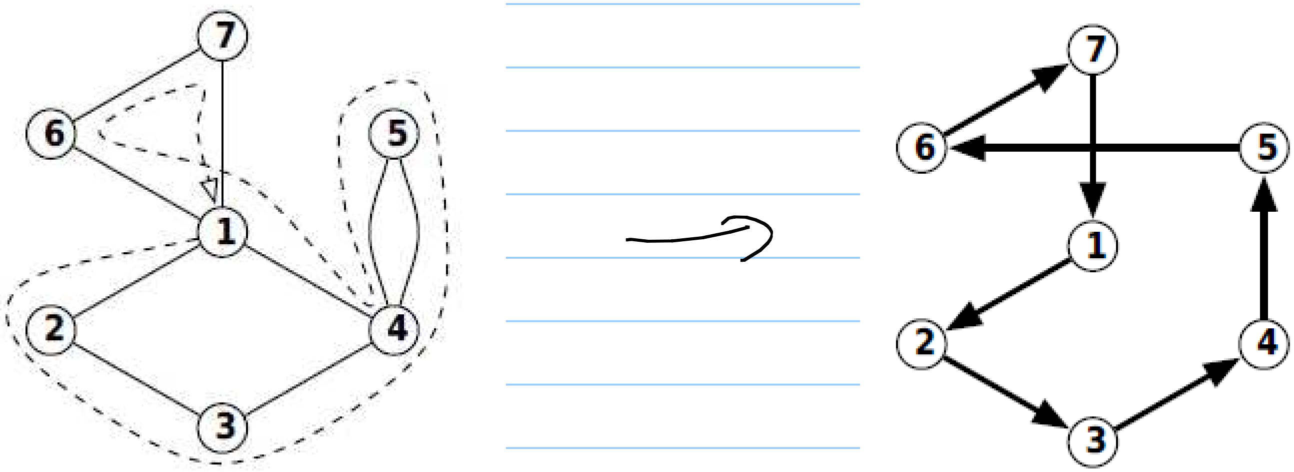
(use every edge in G)





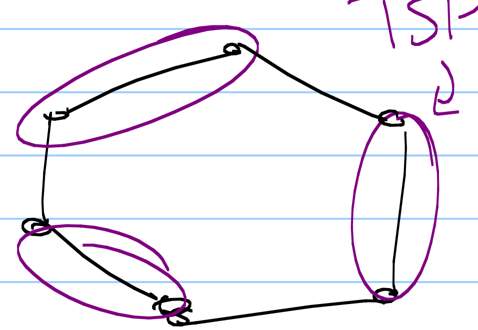
$\underbrace{MST + M}$ $\le$ $OPT + M$

Turn this into a TSP tour.



This yields a $\frac{3}{2}$-approx:

Any tour of O can be separated into 2 matchings:



Smaller has weight $\underbrace{\leq \frac{1}{2} \cdot OPT}$

$\Rightarrow$ minimum matching $\leq \frac{1}{2} \cdot OPT$

$\Rightarrow$ $MST + M \leq OPT + \frac{1}{2} \cdot OPT = \frac{3}{2} \cdot OPT$ $\square$