CS 314 - Graphs

Note Title — 1/22/2010

Announcements

- HW1 up, due next Friday
- A note on office hours...
- HW0 - back on Monday
- Reminder: "I don't know" policy

Graphs - Ch 3

A graph $G=(V, E)$ is a set where

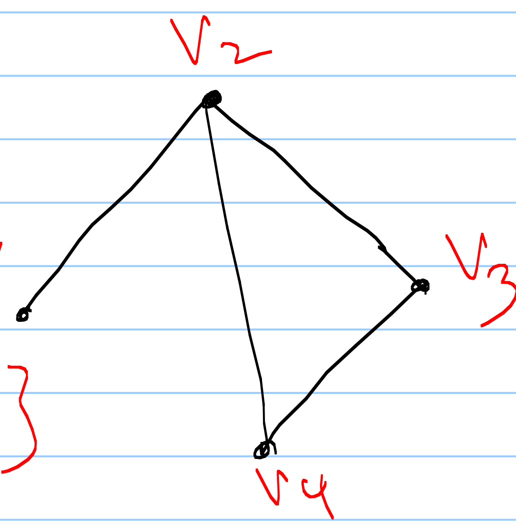
vertices (nodes)

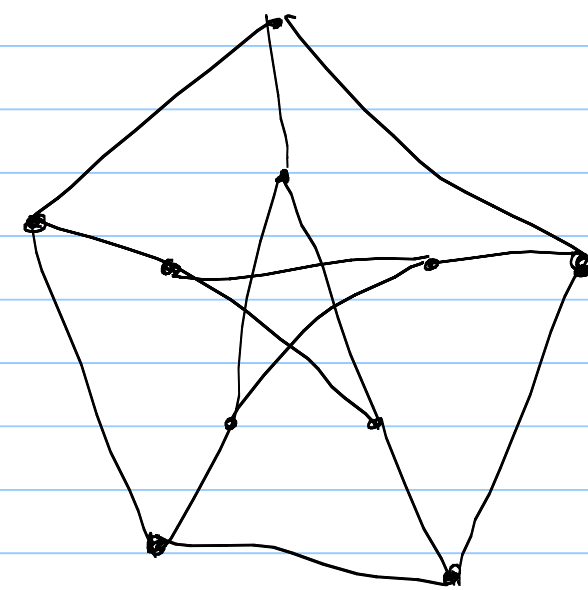
edges (arcs)

$V = \{v_1, \dots, v_n\}$ of vertices

$E = \{ e = \{u, v\} \mid u, v \in V \}$

$E = \{ \{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\} \}$





Examples:

functions

relationships – spatial, files

maps

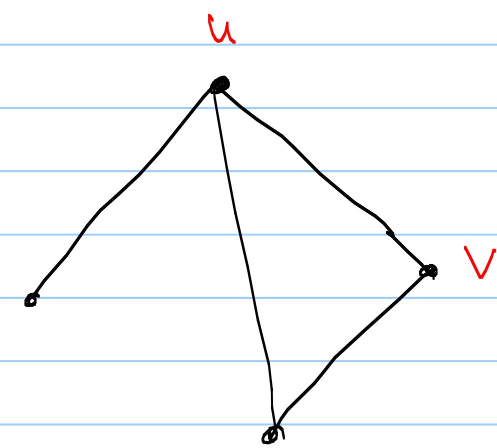
facebook

internet backbone

⋮

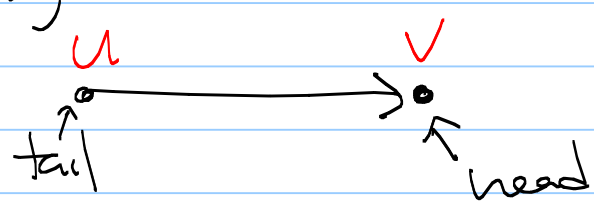
Definitions:

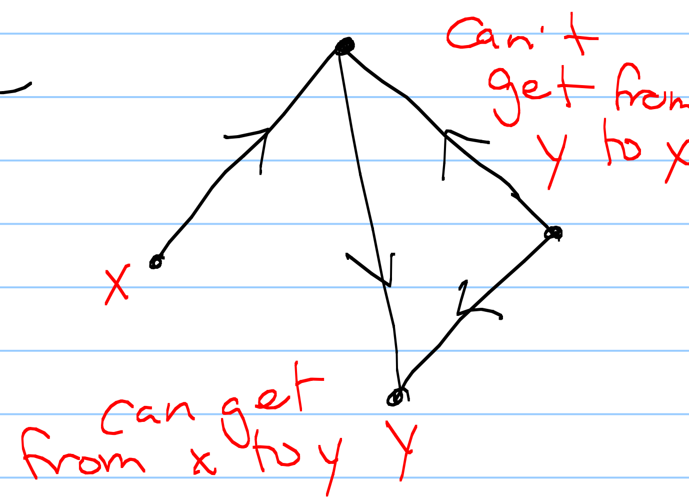
- G is <u>undirected</u> (default) if every edge is an unordered pair, so $\{u,v\} = \{v,u\}$



- G is <u>directed</u> if every edge is an ordered pair

$e = (u,v)$





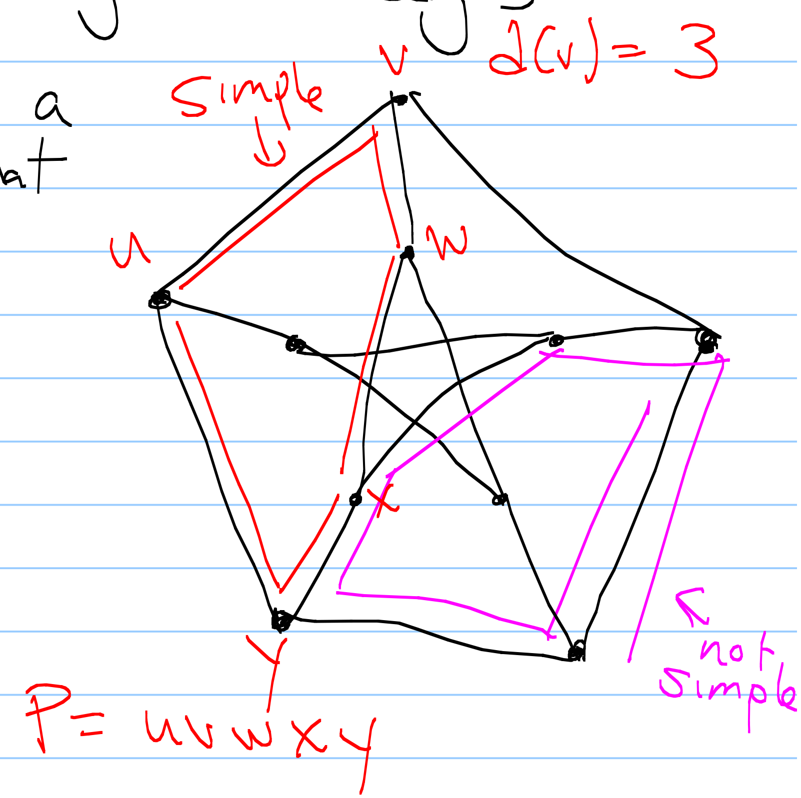
Definitions:

- The **degree** of a vertex v, $d(v)$, is the number of adjacent edges

- A **path** $P = v_1, v_2, \dots, v_k$ is a set of vertices such that $\{v_i, v_{i+1}\} \in E$

- A path is **simple** if all vertices are distinct

- A path is a **cycle** if it is simple except for $v_1 = v_k$



Lemma (degree-sum formula):

$$\sum_{v \in V} d(v) = 2|E|$$

proof: Every edge connects two vertices.

So $|E|$ counts every edge once, but $\sum_{v \in V} d(v)$ counts every edge twice ∎.

Dfns:

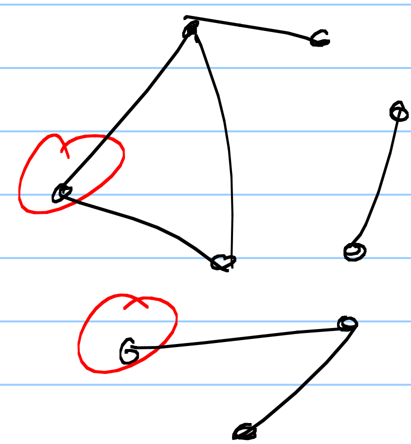
- G is **connected** if $\forall u,v \in V$, $\exists$ a path from u to v

forall — in — there exits

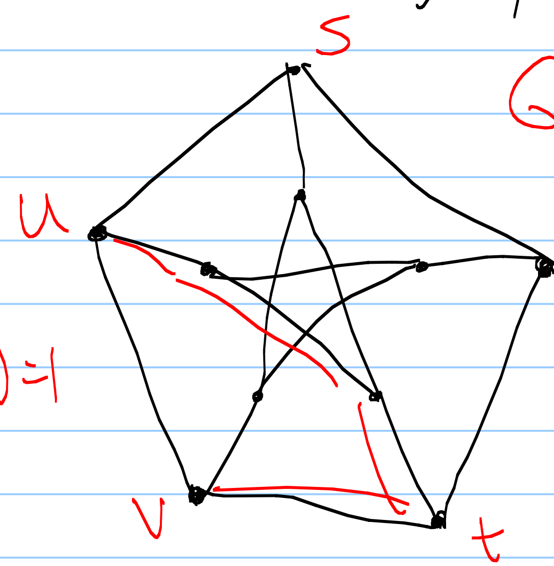
- The distance from u to v, $d(u,v)$, is equal to the length of a minimum u,v path

length = # edges

G1



G2

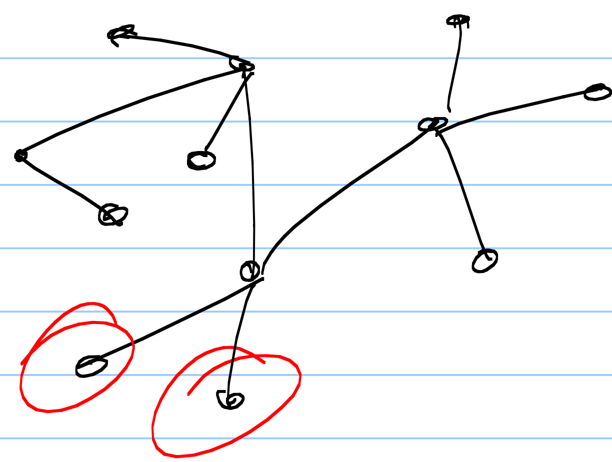


$d(u,v)=1$

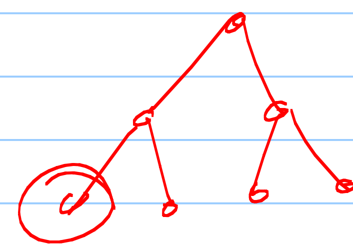
$d(s,t)=2$

Dfn: A tree is a connected, acyclic graph.

no cycles



(no mention of a root!)

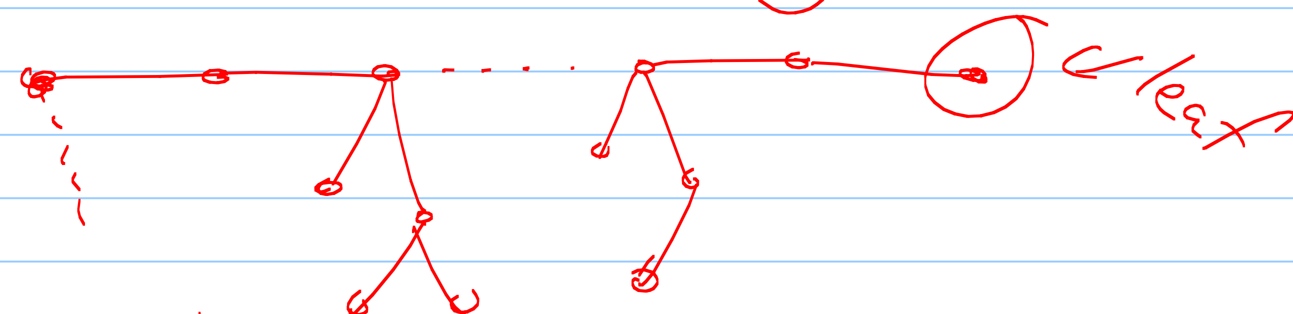


A leaf in a tree is a vertex v with $d(v) = 1$.

Lemma: Every tree with ≥ 2 vertices has a leaf.

pf: A tree is a connect graph with no cycles.

Consider a maximum length simple path, P.



Consider endpts of P. If they have $\deg \geq 2$ then we have a cycle.

Thm: Any 2 of the following imply the 3rd:

a) G is connected
b) G does not contain a cycle
c) G has $n-1$ edges

pf (one of 3 – other 2 are extra credit!)

Show connected, acyclic $\Rightarrow$ $n-1$ edges:

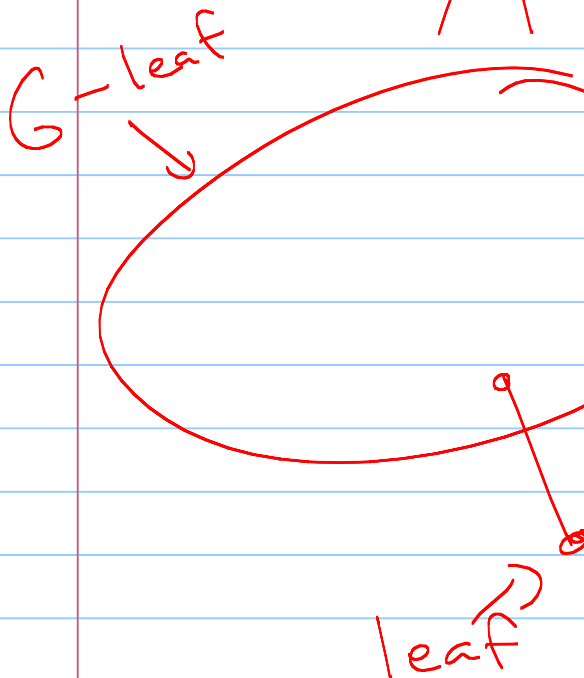
By induction on vertices – so on n:

Base case: • #verts $= 1$ ✓
#edges $= 0$

IH: For a graph with $k < n$ vertices, if it is connected + acyclic then it has $k-1$ edges.

Ind Step: Assume G has n vertices, & is connected & acyclic (ie G is a tree),

By prev. lemma G has a leaf.



Delete the leaf.

G-leaf has 1 less vertex & 1 fewer edge.

G-leaf is acyclic since we haven't added anything

G-leaf is still connected, (since leaf can't be on any simple uv path (for $u,v \neq$ leaf)).

G-leaf has $(n-1)-1$ edges by IH.

$\Rightarrow$ G has $n-1$ edges. $\boxtimes$

Next time:

Algorithms on graphs

Basic question: given 2 nodes, are they connected?