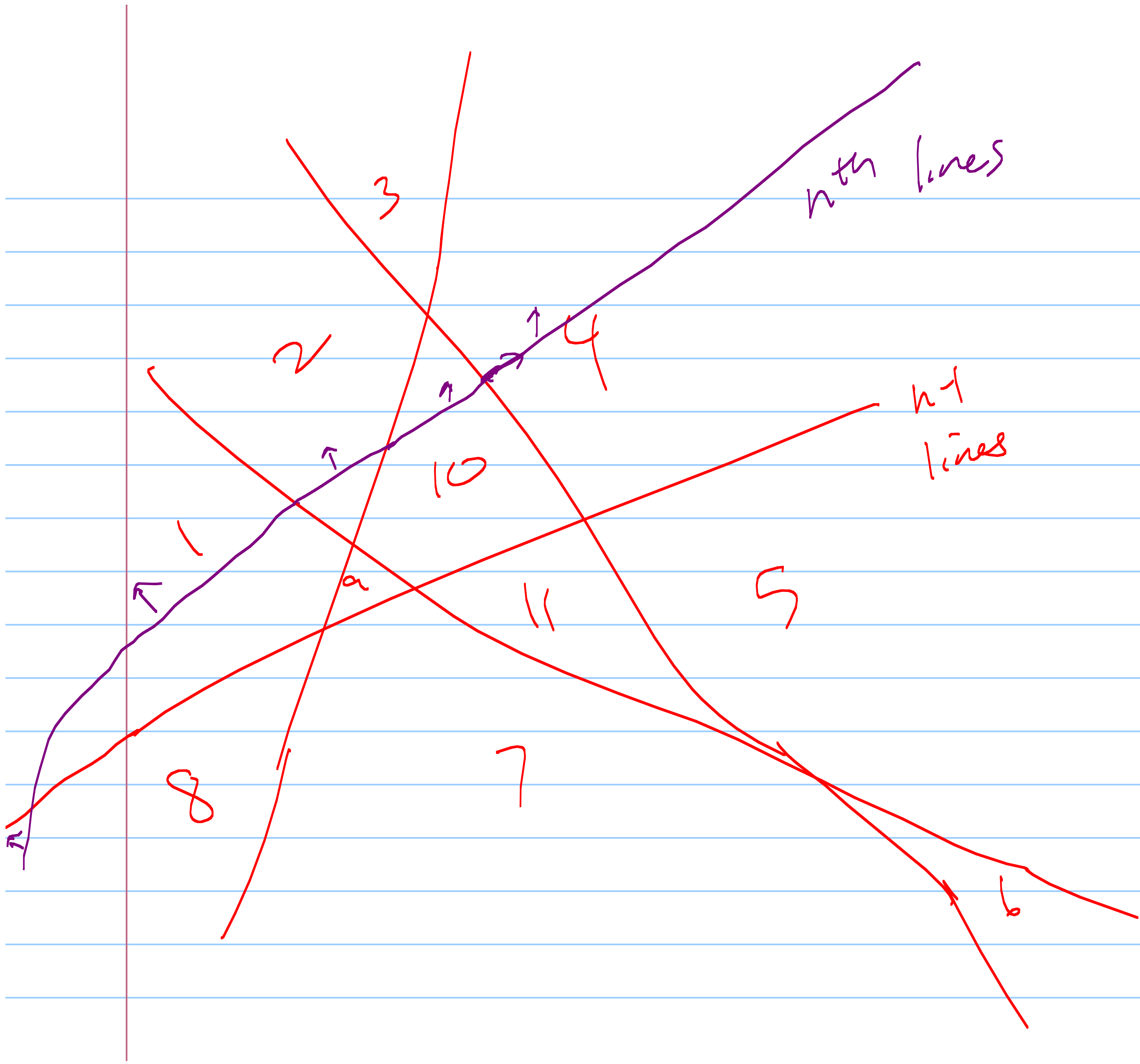
Math 135 - Recursión

Note Title 3/5/2010

<u>Announcements</u>

- HW due Wed. after break
- No class next week
- Reminder about scholarships



2   1 lines

$\downarrow$ +2

4   2 lines

$\downarrow$ +3

7   3

$\downarrow$ +4

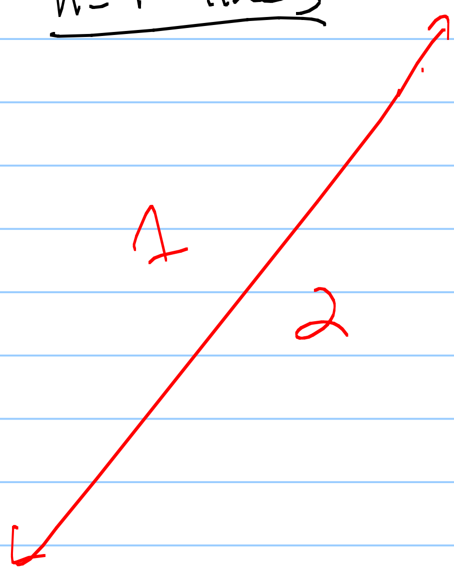
11   4

More Recurrences: Sec 4.3

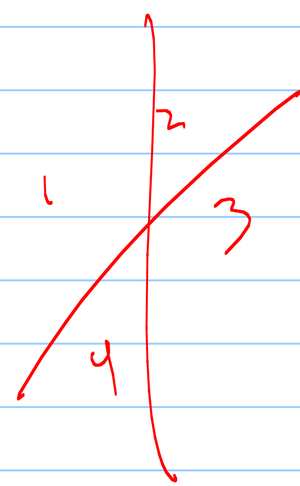
Counting regions in the plane:

Assume:
(no parallel lines)
(no 3 lines at a point)

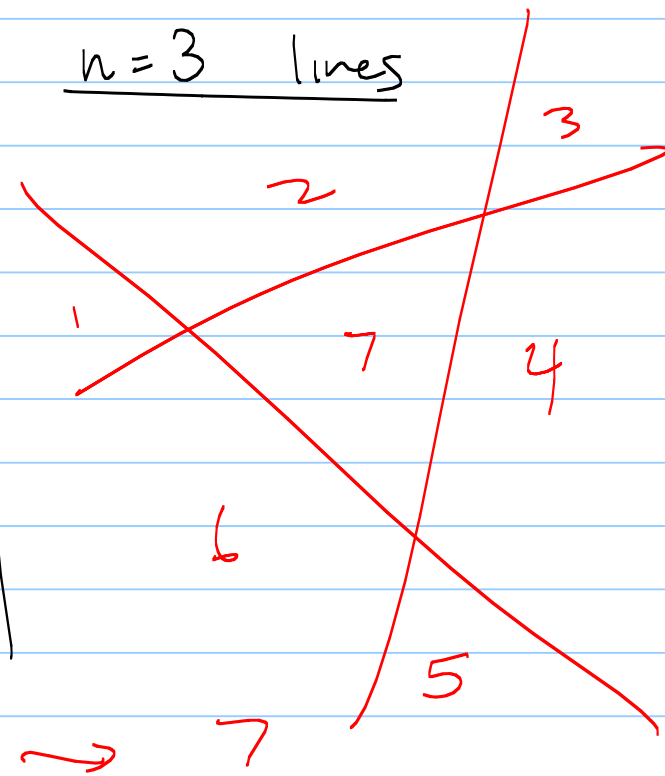
n=1 lines



n=2 lines

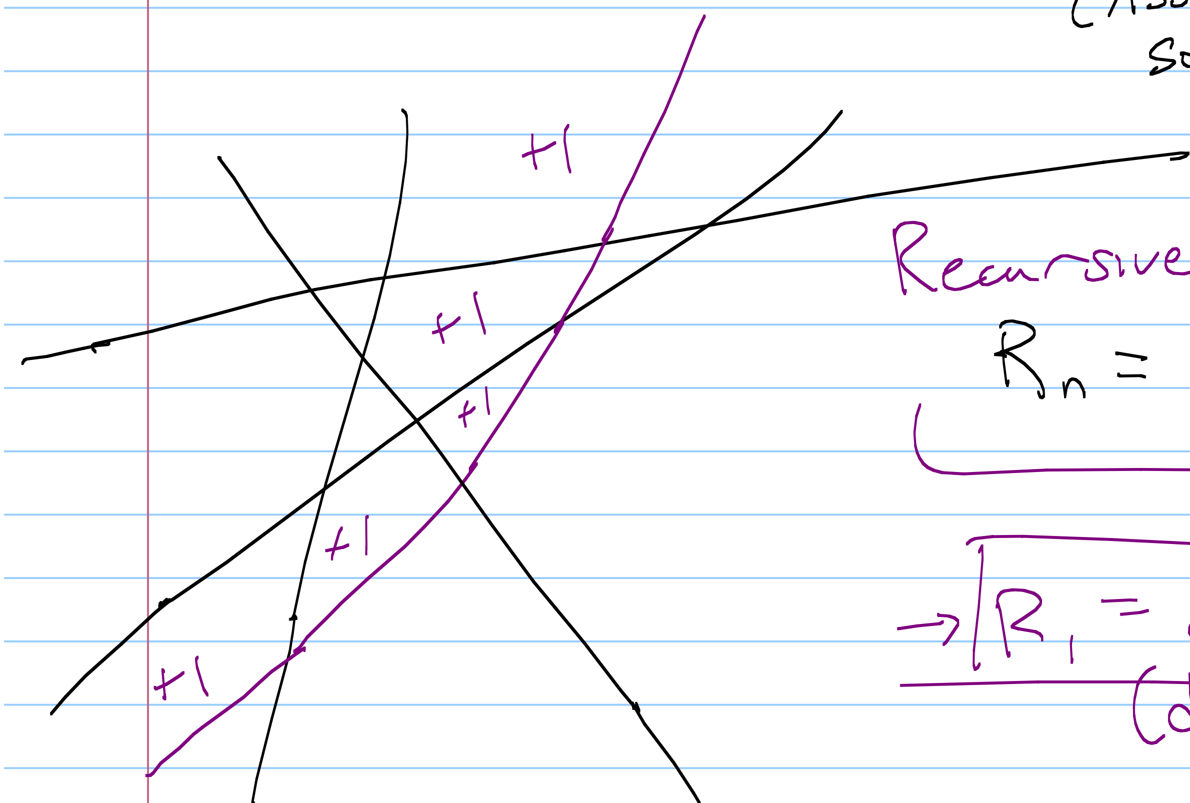


n=3 lines



2 → 4 → 7

Consider $n-1$ lines w/ R_{n-1} regions.
What happens when we add an n^{th} line?
(Assume no parallel lines, so every line crosses every other line)



Recursive definition:

$$\underbrace{R_n = R_{n-1} + n}$$

$$\boxed{R_1 = 2}$$

(don't forget base case!)

$\forall k,\ R_k = R_{k-1} + k$

Closed form for R_n:

Unrolling: $R_n = R_{n-1} + n$

$$= \underbrace{R_{n-2} + (n-1)}_{R_{n-1}} + n$$

$$= R_{n-3} + (n-2) + (n-1) + n$$

$\vdots$

$$= R_1 + 2 + 3 + 4 + \cdots + n$$

$$= 2 + 2 + 3 + \cdots + n$$

$$= 1 + (1 + 2 + \cdots + n)$$

$$= 1 + \frac{n(n+1)}{2}$$

know

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Claim: $R_n = 1 + \frac{n(n+1)}{2}$

pf: induction on n

Base case: $n=1$ $\quad R_1 = 2$

$1 + \frac{1(1+1)}{2} = 1+1 = 2$ ✓

IH: $R_{n-1} = 1 + \frac{(n-1)(n)}{2}$

IS: $R_n = R_{n-1} + n$ (by recursive dfn)

use IH $= 1 + \frac{(n-1)(n)}{2} + n$

$= 1 + \frac{(n-1)(n) + 2n}{2} = 1 + \frac{n((n-1)+2)}{2}$ ∎

Fibonacci

$\boxed{f_n = f_{n-1} + f_{n-2} \quad f_0 = 0,\ f_1 = 1}$

Claim: $f_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$

pf: Induction!

Base cases: $n=0$ $\quad f_0 = 0$

$$\frac{(1+\sqrt{5})^0 - (1-\sqrt{5})^0}{2^0\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

$n=1$: $f_1 = 1$

$$\frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

IH: $\forall k < n$, $f_k = \dfrac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}}$

$x^{n-1} + 2x^{n-2} = x^{n-2}(x+2)$

<u>IS</u>: $f_n = f_{n-1} + f_{n-2}$ by rec. dfn.

apply IH
$$= \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{2}{2}\left[\frac{(1+\sqrt{5})^{n-2} - (1-\sqrt{5})^{n-2}}{2^{n-2}\sqrt{5}}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1} + 2(1+\sqrt{5})^{n-2} - 2(1-\sqrt{5})^{n-2}}{2^{n-1}}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{n-2}(1+\sqrt{5}+2) - (1-\sqrt{5})^{n-2}(1-\sqrt{5}+2)}{2^{n-1}}\right]$$

$\underline{\text{Goal}}$: $f_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{n-2}(1+\sqrt{5}+2) - (1-\sqrt{5})^{n-2}(1-\sqrt{5}+2)}{2^{n-1}}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{n-2}(3+\sqrt{5}) - (1-\sqrt{5})^{n-2}(3-\sqrt{5})}{2^{n-1}}\right]$$

$$(1+\sqrt{5})^2 = (1+\sqrt{5})(1+\sqrt{5}) = 1 + 2\sqrt{5} + 5 = 6 + 2\sqrt{5}$$
$$= 2(3+\sqrt{5})$$

$$\Rightarrow (1+\sqrt{5})^2 = 2(3+\sqrt{5})$$

$$\boxed{\frac{(1+\sqrt{5})^2}{2} = 3+\sqrt{5}}$$

$$x^{n-2}\cdot x^{2} = x^{n}$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{n-2}(3+\sqrt{5}) - (1-\sqrt{5})^{n-2}(3-\sqrt{5})}{2^{n-1}}\right]$$

$$= \frac{1}{\sqrt{5}\cdot 2}\left[\frac{(1+\sqrt{5})^{n-2}\left(\frac{(1+\sqrt{5})^{2}}{\cancel{2}}\right) - (1-\sqrt{5})^{n-2}\left(\frac{(1-\sqrt{5})^{2}}{\cancel{2}}\right)}{2^{n-1}}\right]$$

$$= \frac{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}}{\sqrt{5}\left(2\cdot 2^{n-1}\right)}$$

$\boxtimes$