# Math 135 - Recurrences + Recursion

Note Title 3/3/2010

## Announcements

- HW5 is posted
  due Wed after break

- Scholarships for Math/CS majors
  (for sophomores + juniors)
  applications are in main office (105 RH?)
  (due Wed after break)

# Recurrence Relations

- Use them to model counting problems
- Useful for runtime analysis of recursive algorithms (more next time)
- To solve:
  - unrolling
  - induction
  - more advanced techniques such as Master thm & characteristic eqn method (after break)

## Example:

Consider a sequence of numbers:

| | | | | | |
|---|---|---|---|---|---|
| $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | |
| $\parallel$ | $\parallel$ | $\parallel$ | $\parallel$ | $\parallel$ | $\ldots$ |
| 0 | 2 | 4 | 6 | 8 | |

Closed form

$a_n = 2 \cdot n$

Our goal will be to figure out how to change an induction/recursive dfn into a closed form.

Inductive Dfn:

$a_0 = 0$

$a_n = a_{n-1} + 2$

# Another example:

$$\begin{array}{lcccccccccc} & f_0 & f_1 & f_2 & f_3 & \cdots & & & & & \\ f_n: & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 \ldots \end{array}$$

Recursive Dfn?

$$f_n = f_{n-1} + f_{n-2}$$
$$f_0 = 0$$
$$f_1 = 1$$

Closed form?

$$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

Give recursive dfn of:

① $F(n) = n!$

$\rightarrow F(n) = F(n-1) \cdot n$

② $A(n) = a^n$

$A(n) = a\,A(n-1)$

$G(1) = 1$

$G(n) = n \cdot G(n-1)$

↓ unrolling

$= n \cdot (n-1)\, G(n-2)$

$= n(n-1)(n-2)\, G(n-3)$

$\vdots$

$= n(n-1) \cdots 2 \cdot G(1)$

$= n(n-1) \cdots 2 \cdot 1$

$= n!$

# Compound interest

- $P_0 = \$1,000$ (initial investment)
- make 6% interest per year

How much will you have after $n$ years?

Recursive Dfn: $P_0 = 1,000 \leftarrow$

$$P_n = \underset{\substack{\uparrow \\ \text{amount} \\ \text{from last year}}}{P_{n-1}} + \underset{\substack{\uparrow \\ \text{interest}}}{(.06)P_{n-1}} = 1.06\,P(n-1)$$

Unrolling: Closed form:

$$P_n = (1.06)P_{n-1} = (1.06)(1.06)P_{n-2} = (1.06)^2 P_{n-2}$$

$$= (1.06)^2[1.06 \cdot P_{n-3}] = (1.06)^3 P_{n-3}$$

Closed form ↓ $\qquad\qquad (1.06)^3 P_{n-3}$

<u>Claim</u>: $P_n = (1.06)^n P_0 = (1.06)^n (1,000)$

<u>Pf</u>: (induction on $n$)

<u>Base case</u>: $P_0 \qquad (1.06)^0 \cdot 1000$

$\| \qquad\qquad \|$

$1000 \qquad 1 \cdot 1000 = 1000$ ✓

<u>IH</u>: Assume $P_{n-1} = (1.06)^{n-1} P_0$

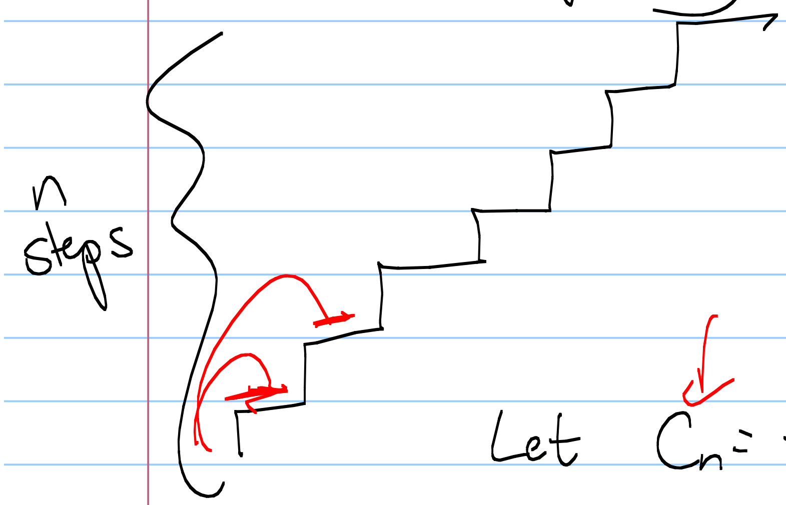
<u>IS</u>: $P_n$ (apply recursive formula)

$= (1.06) P_{n-1}$ & apply IH

$= (1.06)^1 (1.06)^{n-1} P_0$

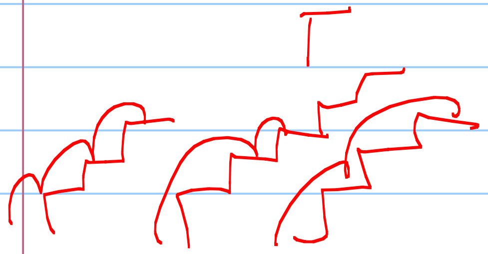
$= (1.06)^n P_0$ ∎

# Stair stepping



If I can take stairs 1 or 2 at a time, how many different ways are there to climb the stairs?

Let $C_n$ = # ways to climb n stairs



$C_1 = 1$
$C_2 = 2$
$C_3 = 3$

either I go 2, or 1 then 1

$$C_n = C_{n-1} + C_{n-2}$$

Think recursively!

First step: We could take 1 or 2 stairs.

- Spps we take 1 stair,

  $C_{n-1}$

- Spps we take 2 stairs.

  $C_{n-2}$

$\Rightarrow C_n = C_{n-1} + C_{n-2}$

+

Base cases: $C_1 = 1$, $C_2 = 2$

Bit strings with no 2 consecutive 0's.

Ex: 1101111

101010

~~10011~~

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{n \text{ bits}}$

Let $b_n$ = # of bit strings w/ no 2 consecutive 0's of length n

Base cases: $b_1 = 2$ $\quad$ 0 or 1

$b_2 = 3$ $\quad$ 11 $\quad$ 10 $\quad$ 01 $\quad$ ~~00~~

Recursive dfn: $\boxed{b_n = ?} = b_{n-1} + b_{n-2}$

$\_\ \_\ \_\ 0\ \_\ \_\ \_\_$

Consider the last bit: $\underbrace{-\ -\ -\ \cdots\ -\ \boxed{\phantom{0}}}_{n \text{ bits}}$ last bit

What could it be?

**Case 1:** could be a 0

$\underbrace{-\ -\ -\ -\ -\ -\ -\ -}_{n-2 \text{ bits}}\ 1\ \boxed{0} \longleftarrow b_{n-2}$

**Case 2:** could be a 1

$\underbrace{-\ -\ -\ -\ -\ -\ -}_{n-1}\ \boxed{1} \qquad b_{n-1}$

# Recursively defined sets

Consider an inductive dfn for a set:

Base step: $3 \in S$

Recursive step: If $x \in S$ and $y \in S$, then $\underbrace{x+y} \in S$.

So what are some elements of $S$?

$\{3, 6, 9, 15, 12, \ldots\}$

Guess? multiples of 3

**Claim:** $S = \overset{A}{\{\text{positive integers divisible by } 3\}}$

**pf:** How do we show 2 sets are equal??

**1:** $A \subseteq S$

Induction on elements of $A = \{3n \mid n \in \mathbb{Z}^+\}$

**BC:** consider smallest pos. int div by 3, $= 3$

is $3 \in S$?

yes, by base step.

**IH:** $\forall k < n$, $3k \in S$

**IS:** $3n = \underbrace{3(n-1)}_{\text{in } S \text{ by IH}} + \underbrace{3}_{\text{in } S \text{ by B.C.}}$

So by recursive dfn, $3(n-1)+3$ is also in $S$ ∎

② Show $S \subseteq A$

given any element $x \in S$, show $x \in A$.

<u>exercise</u>!

(also in text)