Math 135 - Recursion Trees

Note Title 3/19/2010

Announcements

- Homework posted, due next Friday (should be able to do all but last problem already)
- Midterm in 2 weeks (on Wed, 3/31)

Have covered: 4.3, 4.4
Chapter 7 (7.1 & 7.2)

$a_n = c_{n-1} a_{n-1} + \cdots + c_{n-d} a_{n-d} + g(n)$

Method for inhomogeneous recurrences:

1. "Ignore" $g(n)$ and find general solution for the rest
2. Find general solution for $g(n)$
3. Add them together
4. Use base cases (& possibly recurrence) to solve for constants.

How to do it when $g(n) = (\text{polynomial of deg } k) * s^n$?

constant

Ex: $n^2 \cdot 2^n$

Is s a characteristic root?

No

try a general solution of the form $(\text{polynomial of degree } k) \cdot s^n$

Note: use different constants

Yes

what is its multiplicity? Let this be m.

try a general solution of the form $n^m (\text{poly. of degree } k) \cdot s^n$

Ex: $\underbrace{a_n = 5a_{n-1} - 6a_{n-2}} + \overbrace{7^n}^{g(n)}$

(just do general form)

① $x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

roots: 2, 3 $\Rightarrow a_n = c_1 2^n + c_2 3^n$

② $s = 7$ is it a root? NO

$7^n = p(n) \cdot 7^n$ well, $p(n) = 1$, so degree 0

So $(c_3) \cdot 7^n$

③ $\boxed{a_n = c_1 \cdot 2^n + c_2 \cdot 3^n + c_3 \cdot 7^n}$

deg k poly

Ex: $a_n = \underbrace{6a_{n-1} - 9a_{n-2}} + \boxed{n} \cdot 3^n$

s

① $x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0$

roots: 3, with multiplicity $\overset{m}{2}$

$a_n = c_1 3^n + c_2 n 3^n = (c_2 n + c_1) 3^n$

② $s = 3$ that is a root! here, $m = 2$

so $a_n = n^m (\text{poly. deg } k) \cdot 3^n$

$= n^2 (c_3 \cdot n + c_4) 3^n$

③ $a_n = (c_2 n + c_1) 3^n + n^2 (c_3 n + c_4) 3^n$

Ex: $a_n = 6a_{n-1} - 9a_{n-2} + \overbrace{n^2 2^n}^{g(n)}$

① root: 3, with mult. 2

$a_n = (c_1 n + c_2) \cdot 3^n$

② $s = 2$ Not a root!

n^2 is polynomial of deg $= 2$

So $a_n = (c_3 n^2 + c_4 n + c_5) 2^n$

③ $\boxed{a_n = (c_1 n + c_2) 3^n + (c_3 n^2 + c_4 n + c_5) 2^n}$

deg $k=2$

Ex: $a_n = 6a_{n-1} - 9a_{n-2} + \underbrace{(n^2+1)}_{} 3^n$

$5n^2 - 3n$

1. root: 3 w/ multiplicity 2

$a_n = (c_1 n + c_2)3^n$

2. $s=3$, root w/ mult $= \underline{2}$

$a_n = n^2(c_3 n^2 + c_4 n + c_5)\cdot 3^n$

3. $\boxed{a_n = (c_1 n + c_2)3^n + n^2(c_3 n^2 + c_4 n + c_5)\cdot 3^n}$

Ex: $\sum_{i=1}^{n} i = \cancel{\frac{n(n+1)}{2}} = a_n$

Another way to solve — recursion!

$a_n = a_{n-1} + n$, $a_1 = 1$

① $x - 1 = 0$ root: $x = 1$, multiplicity $= 1$

$a_n = c_1 \cdot 1^n$

② $g(n) = n = n \cdot 1^n$ so $s = 1$, root w/ mult $= 1$

$\Rightarrow a_n = n^1 (c_2 n + c_3) \cdot 1^n$

3 $a_n = c_1 \cdot 1^n + n(c_2 n + c_3) \cdot 1^n$

$= c_1 + c_3 \cdot n + c_2 \cdot n^2$

4 $a_1 = 1 = c_1 + c_3 + c_2$

$a_2 = 3 = c_1 + 2c_3 + 4c_2$

$a_0 = 0 = c_1$

so: $c_3 + c_2 = 1 \Rightarrow c_3 = 1 - c_2$

$2c_3 + 4c_2 = 3$ $\quad$ & $c_3 = \frac{1}{2}$

$\Rightarrow 2(1-c_2) + 4c_2 = 3$ so $2c_2 = 1$, so $c_2 = \frac{1}{2}$

$a_n = 0 \cdot 1^n + \frac{1}{2} \cdot n + \frac{1}{2} \cdot n^2 = \frac{n+n^2}{2} = \frac{n(n+1)}{2}$ ✓

Divide + Conquer Recurrences

(Section 7.3 or lecture notes on web)

Remember binary search?

Divided array in half, did one comparison, + recursed in left or right half.

Modeling runtime as a recurrence:

Let $T(n)$ = runtime of binary search on a list of n things

$$T(n) = 5 + T\left(\frac{n}{2}\right)$$

not $n-c$

$$\boxed{T(k) = T\left(\frac{k}{2}\right) + 5}$$

$$T(1) = 1$$

Unrolling:

$$T(n) = T\left(\frac{n}{2}\right) + 5$$

$$= \overbrace{T\left(\frac{n}{4}\right) + 5} + 5$$

$$= T\left(\frac{n}{8}\right) + 5 + 5 + 5$$

$$\vdots$$

$$= \underbrace{T(1)}\ + \underbrace{5 + \cdots + 5}_{\text{how many 5's?}} = 1 + 5\lg n$$

how many 5's? $\leftarrow \lg n$

At each step, divide by 2. Say we do this d times.

$$\frac{n}{2^d} = 1 \Rightarrow n = 2^d \quad \text{(take log of both sides)}$$

$$\boxed{\lg n = d}$$

$$T(k) = 2T\left(\frac{k}{2}\right) + k$$

Merge sort:

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad T(1) = 1$$

$$= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$= 2\left(2\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$\vdots$$

$$1 \, c \, k$$