Math 135 - Some methods of proof

Note Title 1/20/2010

Announcements:

- HW due Friday
  - due at start of class
  - may work in pairs

<u>Last time:</u> introduced proofs

What is a proof?

if p, then q

$p \to q$

T F was a problem

rigorous arguement that something is true

Dfn: n is an <u>even</u> <u>number</u>
$\iff n = 2k$ for some $k \in \mathbb{Z}$

n is an odd number
$\iff n = 2k+1$ for some $k \in \mathbb{Z}$

We showed:

Lemma 1: If n is odd, then n^2 is odd.

direct

If $3n+2$ is odd, then n is odd.

indirect

Thm: If x is even and y is odd, then $x+y$ is odd.

Proof: x is even means $x = 2k$ where $k \in \mathbb{Z}$

y is odd means $y = 2\ell + 1$ where $\ell \in \mathbb{Z}$

Then
$$x+y = 2k + 2\ell + 1 = 2(k+\ell) + 1.$$

Since $k+\ell$ is an integer, this means $x+y$ is odd.

∎

Proof by cases

Thm: For every integer n, n^2+n is even.

pf: n is an integer

Case 1: n is even $\Rightarrow$ $n = 2k$ for $k \in \mathbb{Z}$

then $$n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$$

& since $2k^2+k \in \mathbb{Z}$, means n^2+n is even.

Case 2: n is odd $\Rightarrow$ $n = 2\ell+1$ for $\ell \in \mathbb{Z}$

$$n^2+n = (2\ell+1)^2 + (2\ell+1) = 4\ell^2 + 6\ell + 2 = 2\underbrace{(2\ell^2 + 3\ell + 1)}_{\in \mathbb{Z}}$$

so n^2+n is even $\blacksquare$

__Dfn:__ A real number r is __rational__ if $\exists\, p, q \in \mathbb{Z}$ with $q \neq 0$ such that $r = p/q$.

A real number that is not rational is called __irrational__.

A rational number is in __reduced form__ if p and q have no common divisors.

$\frac{6}{4}$ not reduced, $\frac{3}{2}$ is reduced

Exercise: Prove that the sum of 2 rational numbers is rational.

(How to rewrite as $p \to q$?)

If a and b are rational, then $a+b$ is rational.

pf: $a = \frac{p}{q}$ (since a is rational), $p, q \in \mathbb{Z}$ and $q \neq 0$

$b = \frac{r}{s}$ where $r, s \in \mathbb{Z}$, $s \neq 0$

Then $a+b = \frac{p}{q} + \frac{r}{s} = \frac{ps+rq}{qs}$

$ps + rq \in \mathbb{Z}$ (since $p, q, r, s \in \mathbb{Z}$) and $qs \in \mathbb{Z}$, and $qs \neq 0$ since $q \neq 0$ & $s \neq 0$ $\blacksquare$

Proof by contradiction

A contradiction is a logical statement which is always false.

Suppose we can show [illegible].

If [illegible], then $\neg p$ must also be false. Why?

$\neg p$	q	$\neg p \to q$
T	T	T
T	F	F
F	T	T
F	[illegible]	[illegible]

So if we want to show p is true, one method is:

- assume p is false
- derive a contradiction

(then p must be true)

Called proof by contradiction.

Prove that $\sqrt{2}$ is irrational.

<u>pf by contradiction:</u>

Suppose $\sqrt{2}$ is rational.

$\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$ and assume p/q is <u>reduced form</u>.

So $2 = \frac{p^2}{q^2} \Rightarrow$ [illegible]

Since $q^2 \in \mathbb{Z} \Rightarrow p^2$ is even.

Contrapositive of lemma 1 tells me p is even.

So $p = 2k$ for $k \in \mathbb{Z}$,

$\Rightarrow 2q^2 = (2k)^2 = 4k^2$

$\Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even!

So q & p are both even.

But that means $p = 2k$ & $q = 2m$ for $k, m \in \mathbb{Z}$

$\Rightarrow$ $\frac{p}{q}$ is <u>not</u> in reduced form $\lightning$
(contradicts our assumption)

$\boxtimes$

Thm: Suppose n is an integer.
n is odd $\iff$ n^2 is odd.

if and only if

pf: need to show "if n odd, then n^2 odd"
and "if n^2 odd, then n odd"

already showed this (lemma 1)

Suppose n is even.
$n = 2k$ for $k \in \mathbb{Z}$
So $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

Since $2k^2 \in \mathbb{Z}$, n^2 is even.

$\boxed{\text{Z}}$