Math 135 - Master theorem

Note Title 3/22/2010

Announcements

- HW due Friday
- Midterm 2 next Wed. - review on Monday
- Look for practice midterm later this week

Last time:

$T(k) = 2T\left(\frac{k}{2}\right) + k$

Merge sort:

$T(n) = 2T\left(\frac{n}{2}\right) + n$ $\qquad T(1) = 1$

$= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$

$\rightarrow \quad = \underbrace{2\left(2\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + \frac{n}{2}\right) + n}$

ick

$$\boxed{T(k) = 2T\left(\frac{k}{2}\right) + k}$$

New idea - recursion tree:

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad T(1) = 1 \leftarrow$$

$$T(n) = \sum_{i=0}^{d} (\#\text{ nodes})(\text{work per node})$$

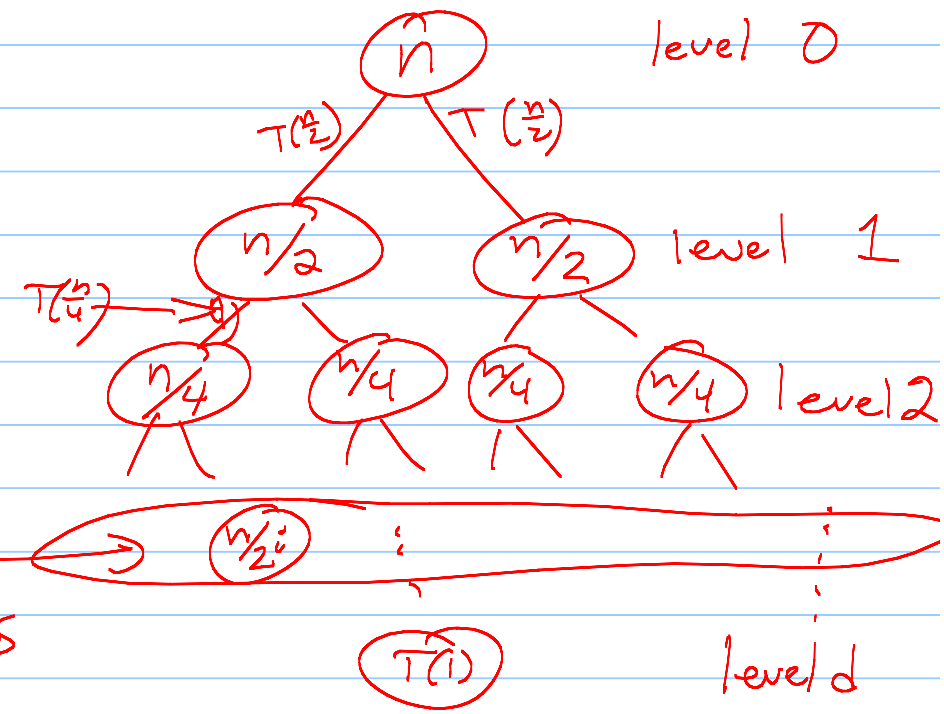
$$= \sum_{i=0}^{d} 2^i \cdot \left(\frac{n}{2^i}\right)$$

$$= \sum_{i=0}^{d} n$$

level i

2^i nodes

Recursion Tree



What is d?

Well $T\left(\frac{n}{2^d}\right) = T(1)$

So $\frac{n}{2^d} = 1$

$n = 2^d$

$\lg n = d$

So: $T(n) = \sum_{i=0}^{\lg n} n = n \sum_{i=0}^{\lg n} 1 = \underbrace{n(\lg n + 1)}$

$= \Theta(n \lg n)$

$$\boxed{S(k) = 3S\left(\tfrac{k}{2}\right) + k^2}$$

Another: $S(n) = 3S\left(\frac{n}{2}\right) + \underline{n^2}$ $\quad S(1) = 1$

$$S\left(\tfrac{n}{2}\right) = 3S\left(\tfrac{n}{4}\right) + \left(\tfrac{n}{2}\right)^2$$

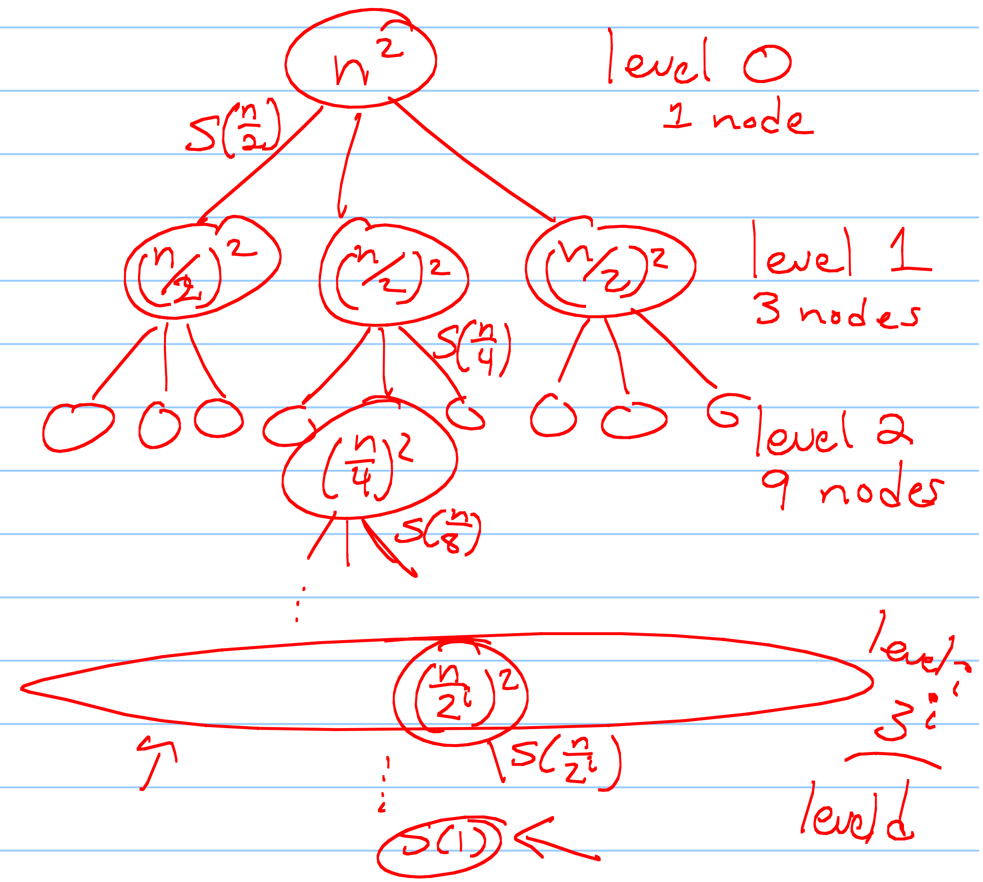
$$S\left(\tfrac{n}{4}\right) = 3S\left(\tfrac{n}{8}\right) + \left(\tfrac{n}{4}\right)^2$$

$$S(n) = \sum_{i=0}^{d} \left(\text{\# nodes on level } i\right)\left(\text{amt of work per node}\right)$$

$$= \sum_{i=0}^{d} 3^i \cdot \left(\frac{n}{2^i}\right)^2$$

What is d?

$$\frac{n}{2^d} = 1 \Rightarrow \underbrace{d = \lg n}$$



$$(2^i)^2 = (2^2)^i$$

$$S(n) = \sum_{i=0}^{\lg n} 3^i \cdot \left(\frac{n}{2^i}\right)^2$$

$$= \sum_{i=0}^{\lg n} 3^i \cdot \frac{n^2}{4^i} = n^2 \cdot \underbrace{\left(\sum_{i=0}^{\lg n} \left(\frac{3}{4}\right)^i\right)}_{c<1} \geq n^2$$

$$\boxed{\text{identity: if } c<1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}}$$

So $S(n) \leq n^2 \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = n^2 \left(\frac{1}{1-\frac{3}{4}}\right) = 4n^2$

$\Rightarrow S(n) = \Theta(n^2)$

$$V(k) = 2V\left(\frac{k}{4}\right) + k^3$$

$$V(n) = 2V\left(\frac{n}{4}\right) + n^3$$

$$V\left(\frac{n}{4}\right) = 2V\left(\frac{n}{16}\right) + \left(\frac{n}{4}\right)^3$$

Solve for d:

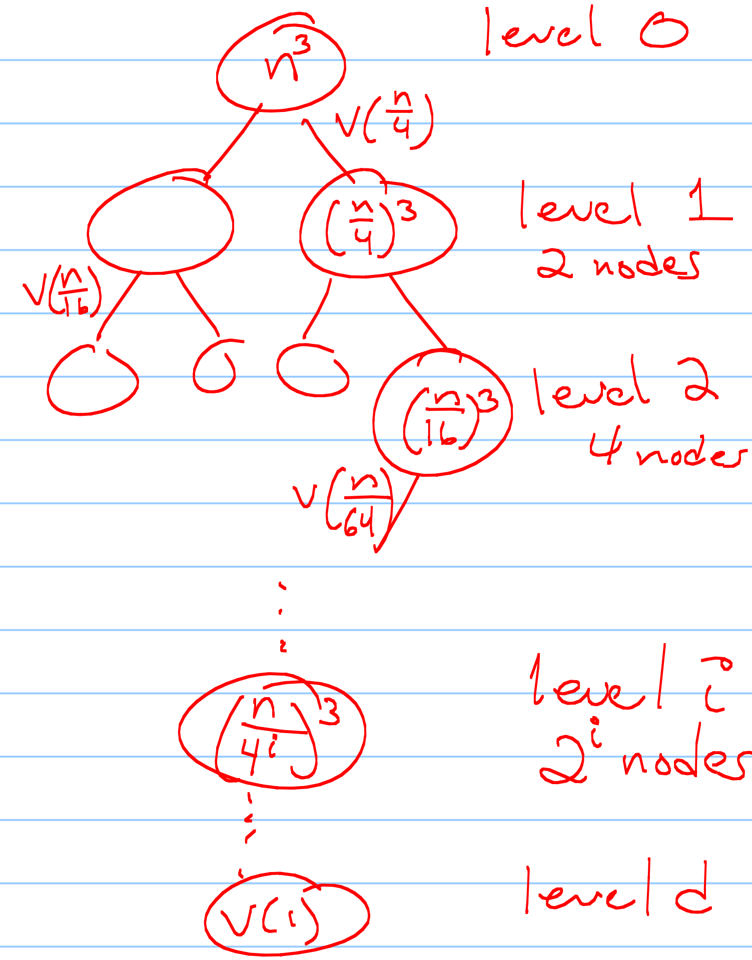
$$\frac{n}{4^d} = 1 \Rightarrow n = 4^d$$

$$\log_4 n = d$$

$$V(n) = \sum_{i=0}^{\log_4 n} 2^i \left(\frac{n}{4^i}\right)^3 = \sum_{i=0}^{\log_4 n} 2^i \cdot \frac{n^3}{64^i}$$

$$= n^3 \sum_{i=0}^{\log_4 n} \left(\frac{2}{64}\right)^i \le n^3 \sum_{i=0}^{\infty} \left(\frac{2}{64}\right)^i$$

$$= n^3 \left(\frac{1}{1-\frac{2}{64}}\right) = O(n^3)$$



Next time: There is a pattern here!

We'll talk about Master theorem:

Let f satisfy $f(n) = a f(\frac{n}{b}) + \Theta(n^d)$,
where $a \geq 1$, b is an integer ≥ 1, and
c and d are real numbers, $c > 0$ & $d \geq 0$.

Then:

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$