Math 135 - Solving recurrences - 7.2

Note Title 3/15/2010

Announcements

- HW is due
- Midterm 2 is in 2 weeks
- HW will be out tomorrow & is due next Friday
- Withdrawal deadline is Friday

  (Come talk to me if you have any questions)
- New Thursday office hours: 9-10 am

Solving Linear recurrences

Dfn: A linear homogenous recurrence has the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}$$

where $c_1, \dots, c_d$ are constants and $c_d \neq 0$.

The order of the recurrence is d.

Examples Yes or no?

$P(n) = 1.06\,P(n-1)$ Yes - order 1

not homogenous

$H(n) = 2H(n-1) + 1$ No

not linear

$R(n) = R(n-1) + (R(n-2))^2$ No

$A(n) = 3A(n-5)$ Yes - order 5

not constant

$C(n) = n\,C(n-1)$ No

$F(n) = F(n-1) + F(n-2)$ - Yes, order 2

Basic Approach

Look for solutions of the form $a_n = r^n$, where r is a constant.

So if $a_n = r^n$ is a solution, have:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

divide by r^{n-k}

$$\underbrace{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0}_{\text{characteristic equation (of the recurrence)}}$$

So:

The sequence $a_n = \{r^n\}$ is a solution

$\Longleftrightarrow$

r is a solution of characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$

[Dfn:
The roots of the characteristic equation are called the characteristic roots]

Ex: $P(n) = 1.06\, P(n-1)$

Char eqn: $x - 1.06 = 0$

Root: $x = 1.06$

$x^2 \quad x^1 \quad x^0$

Ex: $F(n) = F(n-1) + F(n-2)$

Char eqn: $x^2 = x + 1$

$x^2 - x - 1 = 0$ } char eqn

$a = 1$
$b = -1$
$c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

2 roots

Ex: $A(n) = A(n-1) + 2A(n-2)$

char eqn: $x^2 - x - 2 = 0$

$$(x-2)(x+1) = 0$$

roots: $-1, 2$

Ex: $B(n) = 2B(n-1) - B(n-2)$

$$x^2 - 2x + 1 = 0$$

$B(n) = ?$

$$(x-1)(x-1) = 0$$

root: 1, with multiplicity 2

Finding General Solutions

- If r is a non-repeated root of the characteristic equation, then r^n is a solution to the recurrence.

- If r is a repeated root with multiplicity k, then
$$r^n, \; n \cdot r^n, \; \dots, \; n^{k-1} \cdot r^n$$
are all solutions.

- Use linear combinations of these

Ex: $P(n) = 1.06\, P(n-1)$, $P(0) = 10{,}000$

$x - 1.06 = 0$ had root $x = 1.06$

$P(n) = c \cdot (1.06)^n$ (c is constant)

use base case to solve for c:

$$P(0) = \underbrace{10{,}000 = c \cdot (1.06)^0}$$

$$10000 = c(1.06)^0 = c$$

$$\Rightarrow P(n) = (10000)(1.06)^n$$

Ex: $F(n) = F(n-1) + F(n-2)$, $F(0) = 0$, $F(1) = 1$

$x^2 - x - 1 = 0$, roots: $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$,

So $F(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Use base cases to find c_1 & c_2:

$F(0)$: $0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0$

$= c_1 + c_2 \quad \Rightarrow \boxed{c_1 = -c_2}$

$F(1)$: $1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$

$\Rightarrow 1 = (-c_2)\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = \cancel{\frac{-c_2}{2}} - \frac{c_2\sqrt{5}}{2} + \cancel{\frac{c_2}{2}} - \frac{c_2\sqrt{5}}{2}$

$\Rightarrow 1 = -c_2(\sqrt{5}) \Rightarrow c_2 = -\frac{1}{\sqrt{5}}$ So $c_1 = \frac{1}{\sqrt{5}}$

Ex: $B(n) = 2B(n-1) - B(n-2)$, $B(0)=0$, $B(1)=1$

$\rightarrow$ $x^2 - 2x + 1 = 0$ had 1 root, $x = 1$, w/multiplicity 2

so $B(n) = \underbrace{c_1 \cdot 1^n + c_2 \cdot (n 1^n)}$

solve using base cases:

$B(0)$: $0 = c_1 1^0 + \cancel{c_2 (0 \cdot 1^0)} = c_1 \Rightarrow \underbrace{c_1 = 0}$

$B(1)$: $1 = c_1 1^1 + c_2 (1 \cdot 1^1)$

$= c_1 + c_2$

$= 0 + c_2 \Rightarrow c_2 = 1$

Ans: $B(n) = 0 \cdot 1^n + 1 \cdot (n \cdot 1^n) = n$

Ex: Spps we get char eqn for $C(n)$ as:

$$(x-2)^3(x-5)^2(x-9)=0$$

What is form of the general solution?

roots: 2 with multiplicity 3

5 with multiplicity 2

9 (mult = 1)

$$C(n) = c_1 \cdot 2^n + c_2(n2^n) + c_3(n^2 2^n) + c_4 \cdot 5^n + c_5(n5^n) + c_6 \cdot 9^n$$

Linear

Dfn: Inhomogeneous recurrences have an added function $g(n)$:

$$f(n) = c_1 f(n-1) + \cdots + c_d f(n-d) + g(n)$$

Ex: $F(n) = F(n-1) + F(n-2) + 1$

$$A(n) = 4A(n-1) + 3^n$$

Method for inhomogeneous recurrences:

1. "Ignore" $g(n)$ and find general solution for the rest
2. Find general solution for $g(n)$
3. Add them together
4. Use base cases (& possibly recurrence) to solve for constants.

We'll talk about how to do step 2 when

$$g(n) = \underbrace{(\text{polynomial of degree } k)}\cdot \underbrace{s^n}$$

(where s constant)

Ex: $g(n) = \underbrace{(n^2+1)\cdot 2^n}$ $s=2$

$g(n) = \underbrace{(n+5)}\cdot 1^n$ $s=1$

need to know s^n

How to do it:

Is s a characteristic root?

No → try a general solution of the form $(\text{polynomial of degree } k) \cdot s^n$

use different constants

Yes → what is its multiplicity? Let this be m

→ try a general solution of the form $n^m (\text{poly. of degree } k) \cdot s^n$

Ex: $f(0) = 1$

$f(n) = 4f(n-1) + 3^n$

$f(1) = 4f(0) + 3^1 = 7$

① char eqn of homogenous part (ignore 3^n)

$f(n) = 4f(n-1)$

$x - 4 = 0$

root: 4

$f(n) = c_1 4^n$

② $g(n) = 3^n \Rightarrow c_2 \cdot 3^n$

③ $\boxed{f(n) = c_1 4^n + c_2 3^n}$ general form

④ $1 = c_1 4^0 + c_2 3^0 = c_1 + c_2$

$7 = c_1 \cdot 4 + c_2 \cdot 3$

$$c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2$$

$$4c_1 + 3c_2 = 7$$

$$4(1 - c_2) + 3c_2 = 7$$

$$4 - 4c_2 + 3c_2 = 7$$

$$-c_2 = 3$$

$$c_2 = -3$$

$$\Downarrow$$

$$c_1 = 1 - (-3) = 4$$

<u>SO</u>: $\boxed{f(n) = 4 \cdot 4^n - 3 \cdot 3^n}$