Math 135 - Infinite Sets

Note Title 2/8/2010

Announcements

- HW due Friday
- Office hours tomorrow 9-10am
- Midterm next Wed.

Infinite Sets (Ch. 2.4, end of section)

Dfn: Two sets A & B have the same cardinality if and only if there is a bijection from A to B.

Thm: $\mathbb{N}$ and $\mathbb{Z}$ have same cardinality.

Pf: dovetailing

$$\begin{array}{ccc}
\mathbb{N} & & \mathbb{Z} \\
0 & \longrightarrow & 0 \\
1 & \longrightarrow & 1 \\
2 & \longrightarrow & -1 \\
3 & \longrightarrow & 2 \\
4 & \longrightarrow & -2 \\
\vdots & & \vdots
\end{array}$$

this is a bijection

rational #s

Thm: $\mathbb{N}$ and $\mathbb{Q}$ have same cardinality.

pf (picture)

numerator

denom.

	0	1	2	3	4	5	...
1	$0/1$	$1/1$	$2/1$	$3/1$	$4/1$	...	
2	$0/2$	$1/2$	$2/2$	$3/2$	...		
3	$0/3$	$1/3$	$2/3$	$3/3$	...		
4							
5							
6							
⋮							

p/q ✓

Are there sets which are "bigger" than $\mathbb{N}$?

<u>Dfn</u>: A set A is <u>countable</u> if there is a bijection $f: \mathbb{N} \to A$ (or if A is finite).

Q: are there <u>uncountably</u> infinite sets?

$\mathbb{R}$

$P(\mathbb{N})$

Thm: $\mathbb{R}$ is not countable.

Actually, we'll show $(0,1) \subseteq \mathbb{R}$ is not countable.

Called Cantor diagonalization arguement:

pf: Look at decimal representations of numbers $\in (0,1)$.

Proof by contradiction:

Suppose there is a bijection $f: \mathbb{N} \to \mathbb{R}$.

Consider that function as giving a list of all decimal ~~#s~~ between 0 + 1.

1st $\quad .\ x_{11}\quad x_{12}\quad x_{13}\quad x_{14}\ \cdots$

2nd $\quad .\ x_{21}\quad x_{22}\quad x_{23}\quad x_{24}\ \cdots$

3rd $\quad .\ x_{31}\quad x_{32}\quad x_{33}\quad x_{34}\ \cdots$

$.\ x_{41}\quad x_{42}\quad x_{43}\quad x_{44}\ \cdots$

$\vdots$

$.0\,1\,5\,3\ldots$

$.3\,2\,5\,9\ldots$

$.6\,1\,2\,4\ldots$

$.2\,4\,3\,9$

$.1\,3\,3\,0\ldots$

Construct another decimal #:

$$d = .d_1\ d_2\ d_3\ d_4\ d_5\ \cdots$$

let $d_i = x_{ii} + 1$

So $d_i \neq x_{ii}$ for all i

So d is <u>different</u> from every row of my listing!

But why do we care??

Well, we care about <u>computable</u> things.

How many computer programs are there?

So a computer program can be "just" a number!

How many functions from $\mathbb{N} \to \{0, 1\}$ are there?

x	0	1	2	3	4	...
$f(x)$						

these look like fractions b/t 0 and 1

$\Rightarrow$ there are a <u>lot</u> of uncomputable functions!