Math 135 - Graphs (part 2)

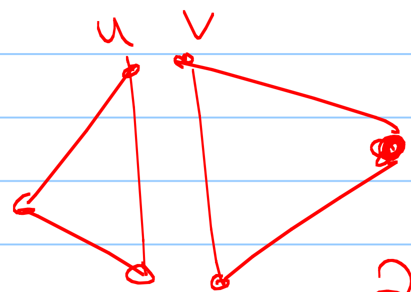
Note Title 4/21/2010

Announcements

- HW due Friday
- Next HW up tomorrow or Friday, due on last day of class

Dfn: A graph G is connected if for every pair of vertices u + v, there is a u-v walk in G.

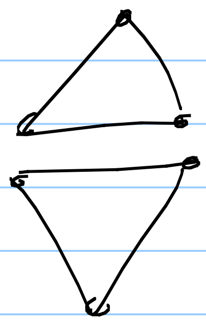
The components of G are maximally connected subgraphs.



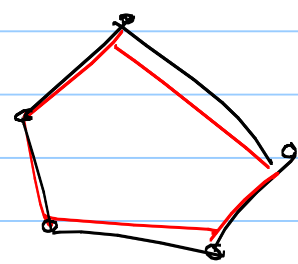
disconnected

2 components

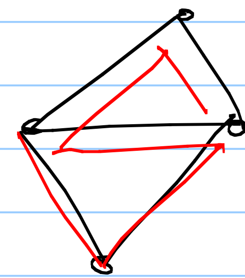
<u>Dfn</u>: An <u>Eulerian</u> circuit is a circuit which uses every edge exactly once.



No



Yes



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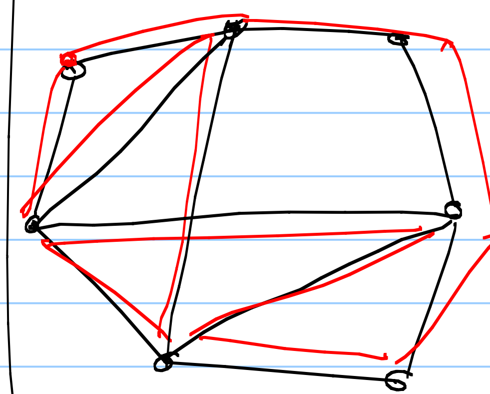
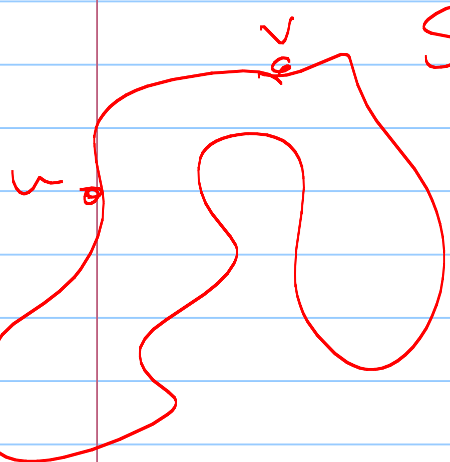


Yes

What graphs have these?

Thm: A graph G has an Eulerian circuit if + only if G is connected + every vertex has even degree.

pf: **$\Rightarrow$):** Suppose G has Eulerian circuit.



Show G is connected: Take $u, v \in V(G)$. Know u + v appear on our circuit. So the circuit gives us the u-v walk.

Show every vertex has even degree: Consider a vertex v. Walk along the circuit. Every time we visit v, we know 2 more edges adjacent to v, so we add $+2$ to $d(v)$. $\Rightarrow$ in the end, $d(v)$ is even.

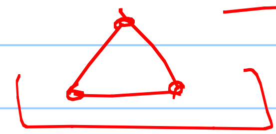
$\Leftarrow$: G is connected & every vertex has even degree. Show an Eulerian circuit.

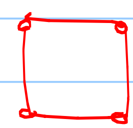
↳ $\forall v \in V$, $d(v) \geq 2$ (since if $d(v) = 0$, then G is not connected)

G has a cycle (by prev. Thm)

Induction on # of vertices, n:

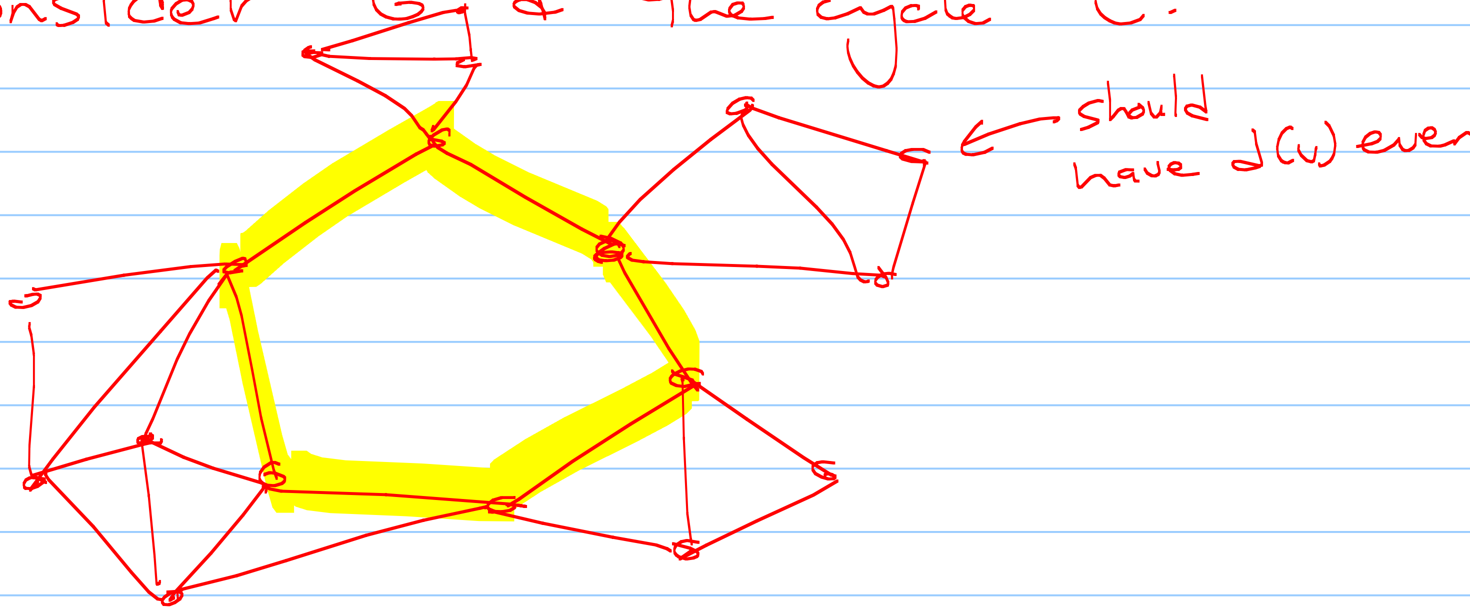
Base case: $n = 3$





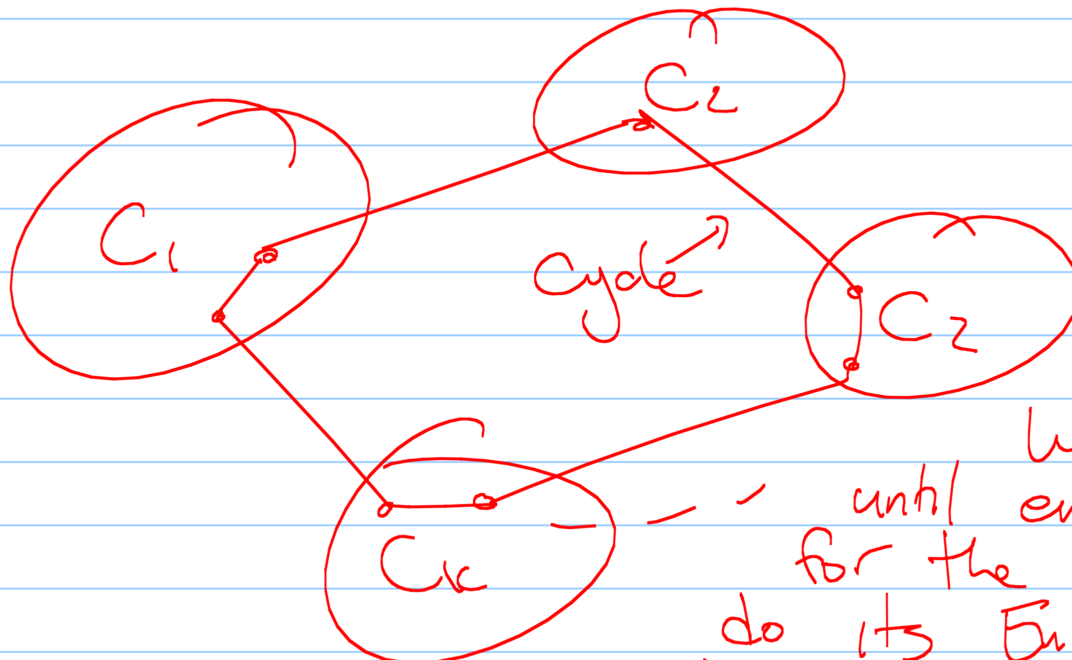
IH: For graphs on < n vertices which are even & connected, we can find an Eulerian circuit.

IS: Consider G & the cycle C:



delete my cycle. Left with some number of components.
Consider 1 component. Every vertex in it still has even degree!

So each component (by my IH) has its own Euler circuit.



Use these to make a Euler circuit for G.

Start at a vertex on the cycle. Walk along cycle until enter a component for the first time, then do its Euler circuit. That returns to same vertex, & we continue along the cycle. ⊠

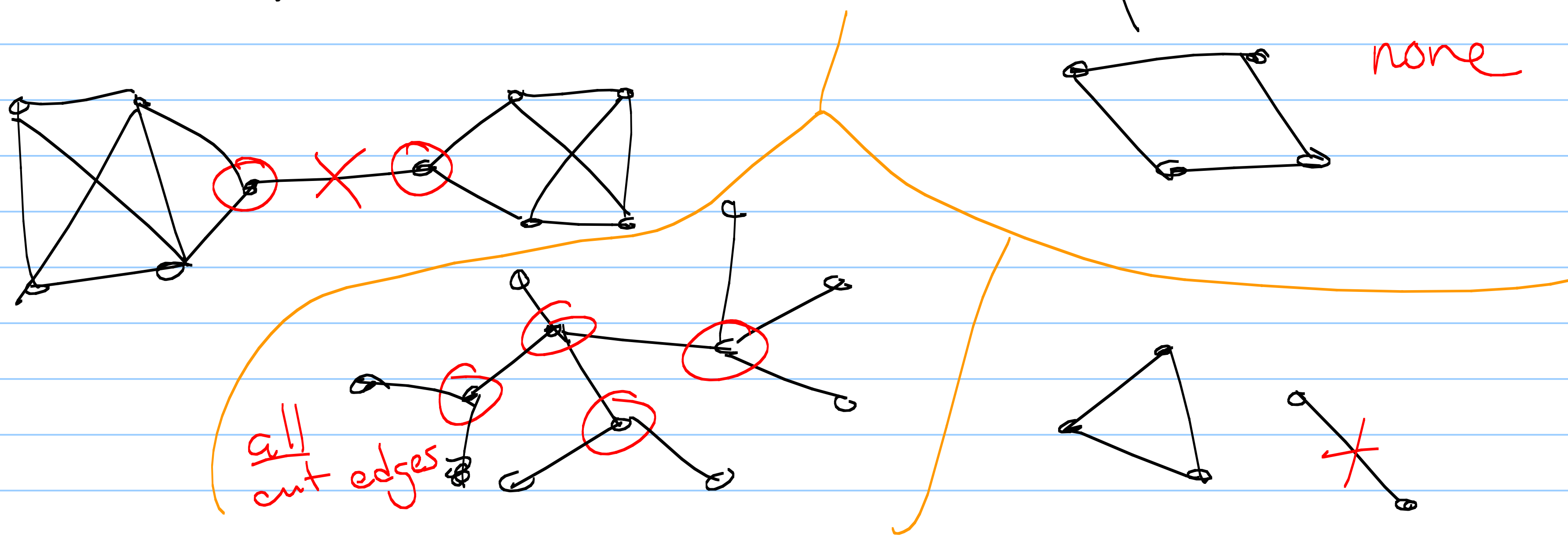
Thm: Every uv-walk contains a uv-path.

pf: Induction on the length of the walk.

leave for worksheet...

Dfn: A *cut-edge* in a graph is an edge whose deletion increases the number of components.

A *cut-vertex* is a vertex whose deletion increases the # of components.

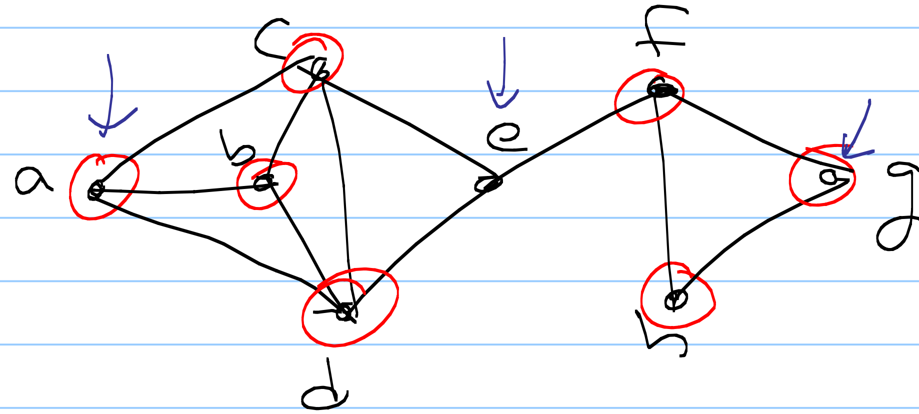


Thm: An edge is a cut edge $\iff$ it does not belong to any cycle.

pf: In worksheet next time.
(or HW?)

Dfn: In a graph G, a clique is a set of vertices that are pairwise adjacent.

An independent set is a set of vertices that are pairwise non-adjacent.



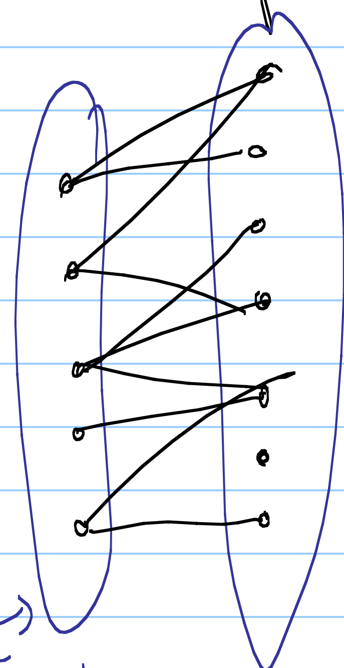
clique - f, g, h

{a b c d}

Ind set:
{g, e, a}

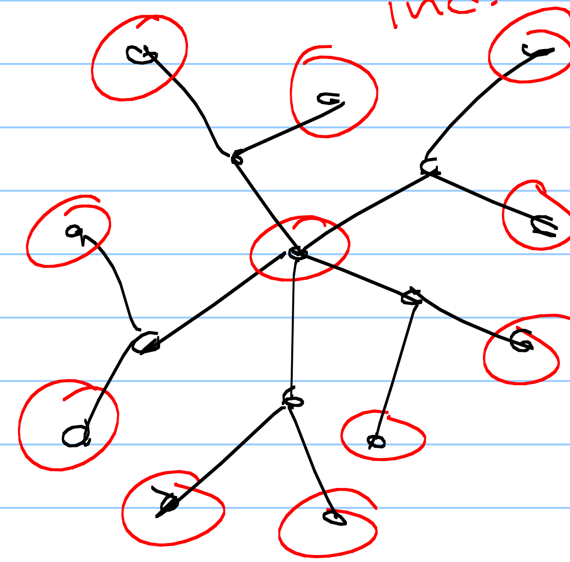
Dfn: A graph G is bipartite if the vertices in G can be partitioned into 2 independent sets.

Ex:

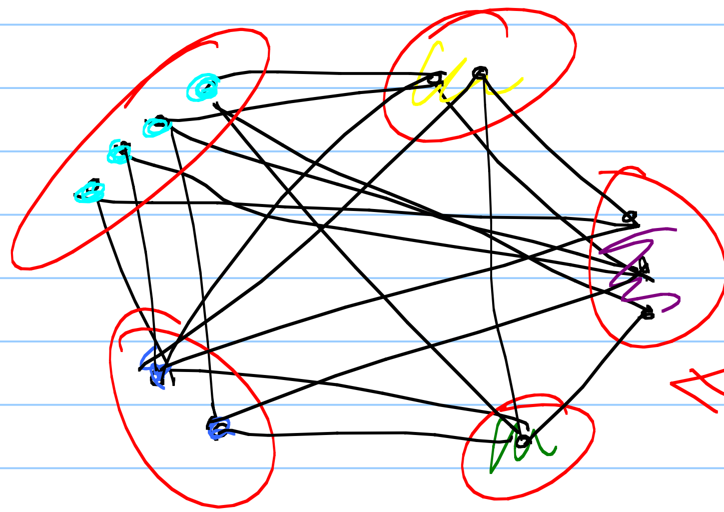


pairwise non-adjacent

red is one ind. set.



Dfn: A graph is k-partite if its vertices can be partitioned into k independent sets.



5-partite

bipartite
$\Updownarrow$
2-partite

Colorability

A graph is k-colorable if we can color each vertex with one of k colors so that adjacent vertices get different colors.

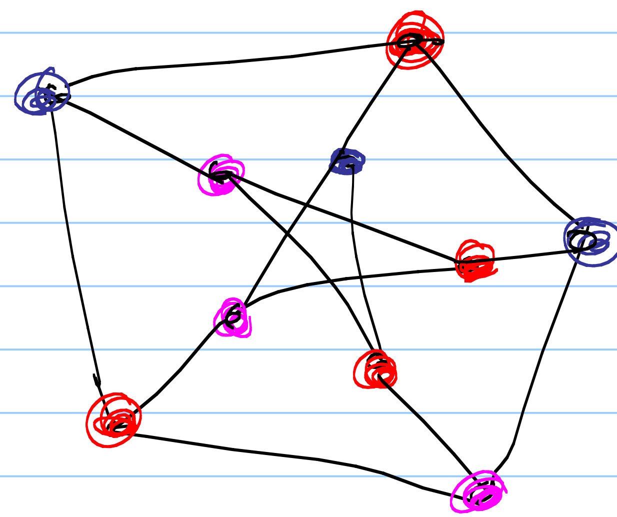
Thm: G is k-partite $\iff$ G is k-colorable.

pf: $\Rightarrow$: k-partite means we can divide G into k independent sets.
Color each independent set 1 color.
Since all edges go between the sets, no edge has endpts of the same color.

$\Leftarrow$: Each color class defines an independent set.

Dfn: The chromatic number of a graph is the minimum k s.t. G can be k-colored.

(Written $\chi(G)$.)



3-colorable

not 2-colorable

so $\chi(G) = 3$

Cor: G is bipartite $\iff \chi(G) \le 2$.

Why? using prev. thm

k-partite $\iff$ k-colorable.