Math 135 - Graph Theory

Note Title 4/19/2010

Announcements

- HW due Friday
- Next HW - out Wednesday
  - last one! Will be due last day of class
- Final is May 7, 2-4pm
  (Let me know if you have a conflict by the end of this week)

Graphs Ch 9

Motivation: Model relationships or connections

- Cities & roads
- Internet Connectivity
  (routes, computers, etc.)
- Webpage links
- Social Networks
- Biological Networks

  ⋮

Dfn: A graph $G=(V,E)$ is a pair of sets:

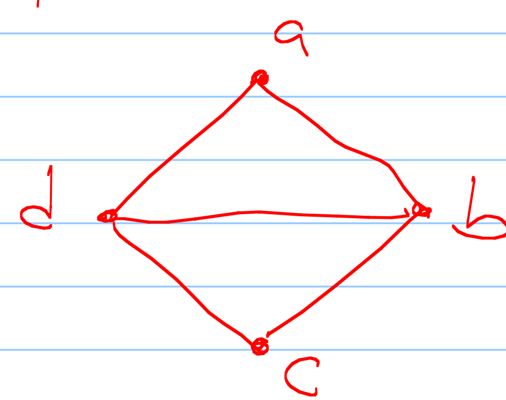
- V is a set of vertices
- E is a set of edges

Each edge is associated with 2 vertices, called its **endpoints**.

$e \in E \Rightarrow e = \{u\,v\}$

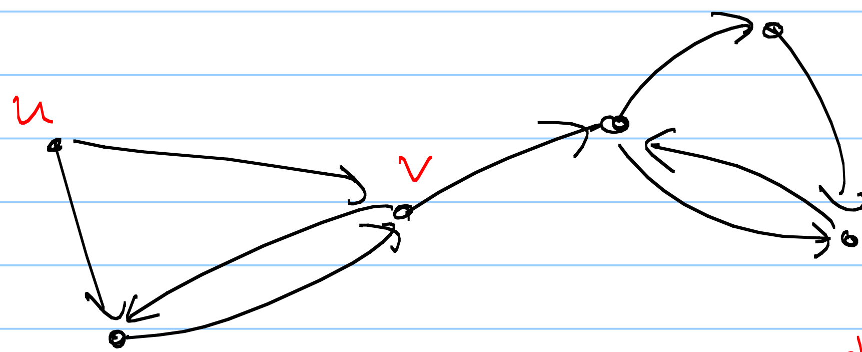
Ex: $V = \{a, b, c, d\}$

$E = \{ \{ab\}, \{bc\}, \{cd\}, \{ad\}, \{bd\} \}$



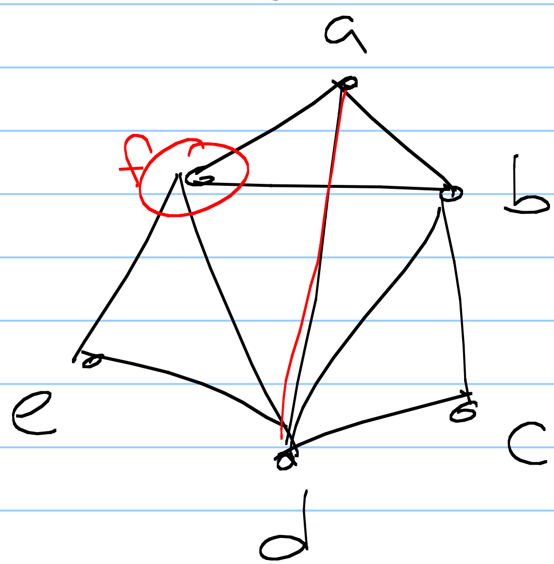
In a **directed** graph, each edge is an **ordered** pair - not just a set.

Ex: $e = \overrightarrow{uv}$



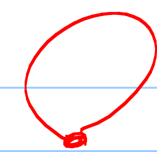
one way streets

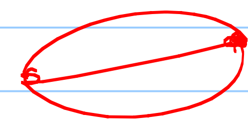
Dfn: We say an edge is incident to its endpoints, & two vertices are adjacent if there is an edge between them.



Ex: a & d are adjacent

f is incident to 4 edges

We can have loops: 

or multiple edges: 

A graph is called simple if it has no loops or multiple edges.

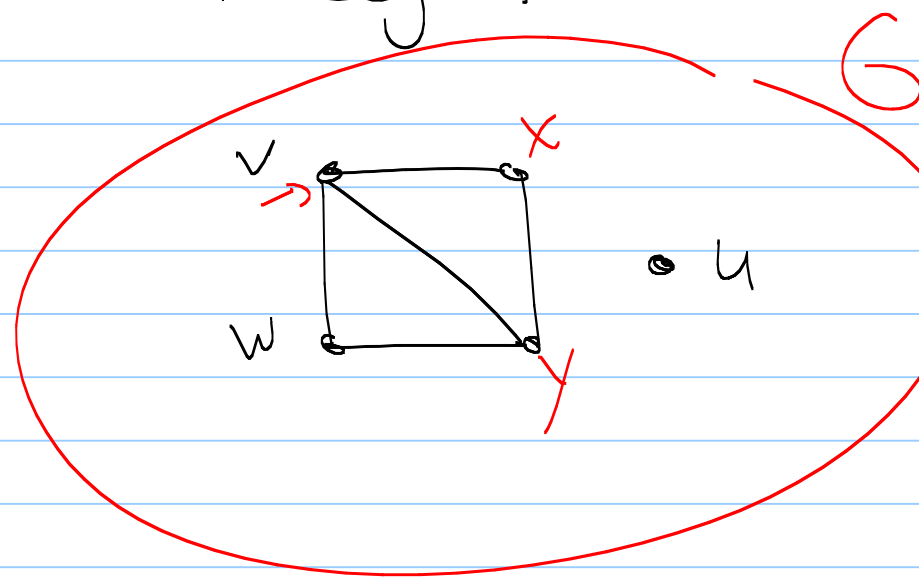
We'll (usually) deal with simple, undirected graphs here.

Dfn: The <u>degree</u> of a vertex, $d(v)$, is the number of incident edges.

$d(v) = 3$

$d(w) = 2$

$d(u) = 0$

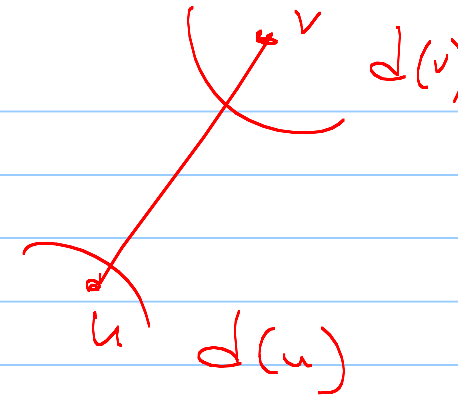


$3+2+3+2+0 = 10$

Sum over all vertices of the degree of each vertex

Thm: $$\sum_{v \in V} d(v) = 2|E|$$

Degree Sum formula, or Handshaking Lemma?

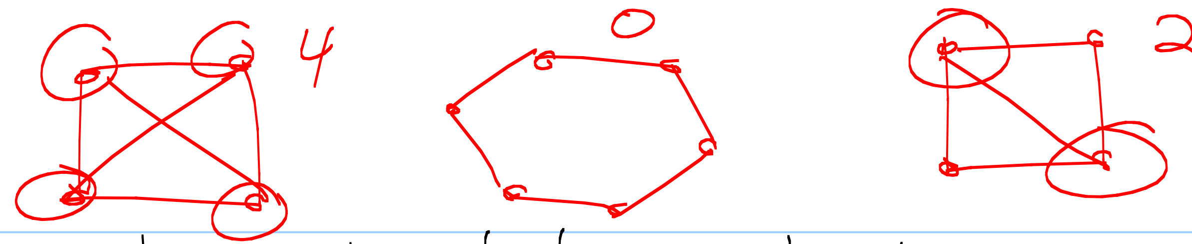


pf: combinatorial proof:

Right side: count every edge twice

Left side: Every edge has 2 endpoints, so each edge counts in d(v) for 2 different vertices.

So each edge is counted twice in the left side.



Thm: In a simple, undirected graph, the number of nodes with odd degree is even.

pf: Degree sum formula:

$$\sum_{v \in V} d(v) = 2|E| \leftarrow \text{even}$$

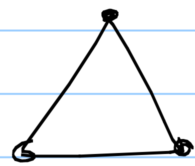
Recall: even + odd = odd
odd + odd = even
even + even = even

Spps have an odd number of vertices of odd degree. They <u>must</u> sum to an odd number.
The even degrees sum to an even number.

So this would result in an odd number

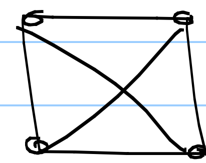
Some special classes of graphs:

- Complete graph K_n (on n vertices):

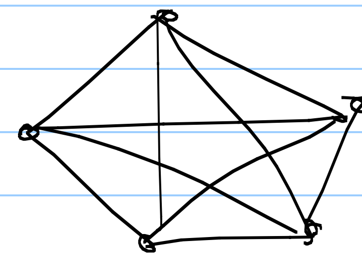


K_3

3 edges



K_4 6 edges



K_5

10 edges

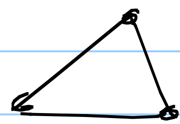
Q: How many edges does K_n have?

$$\binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

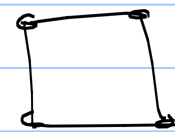
- Cycles C_n

  n vertices $v_1, v_2, \dots, v_n$

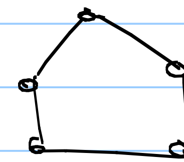
  & n edges $\{v_i v_{i+1}\}$ + $\{v_n v_1\}$



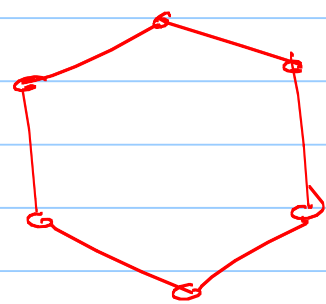
C_3



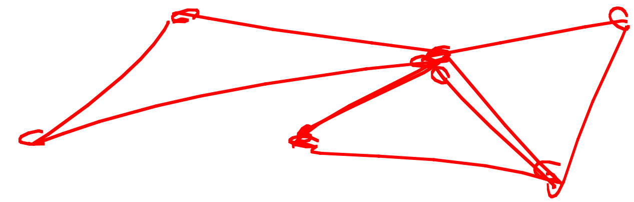
C_4



C_5 . . .



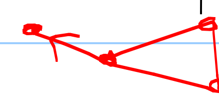
C_6



Dfn: A *walk* is a sequence:
$v_0 e_1 v_1 e_2 v_2 e_3 v_3 \dots v_k$ of alternating vertices & edges where each edge e_i is incident to v_{i-1} and v_i.

It is *closed* if $v_0 = v_k$.

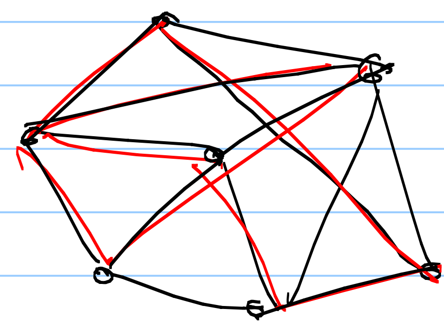
A *trail* is a walk with no repeated edges.



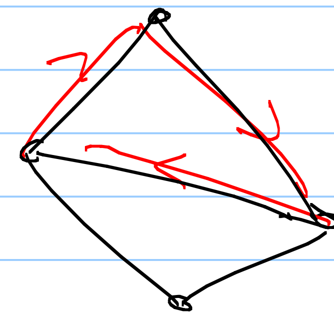
A *path* is a trail with no repeated vertices.

Dfn: We call a closed trail a circuit.

We call a closed path a cycle.



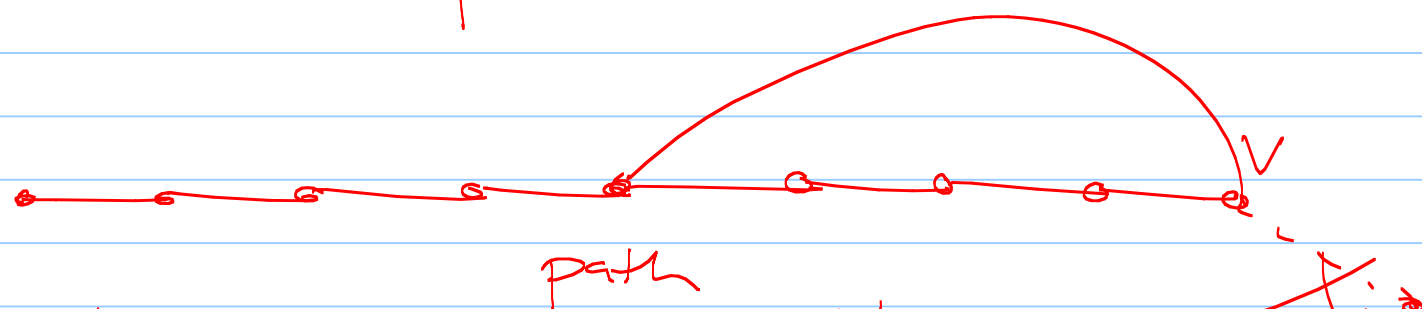
circuit



cycle

<u>Prop:</u> If every vertex in G has degree ≥ 2, then G contains a cycle.

<u>pf:</u> Consider a path that is maximal.



(Can't make this longer, b/c it is maximal)

Consider endpoint v. v can't connect off the path, b/c that would mean path could get longer. But $d(v) \geq 2$, so v must have a neighbor <u>on</u> the path.

So this gives a cycle.

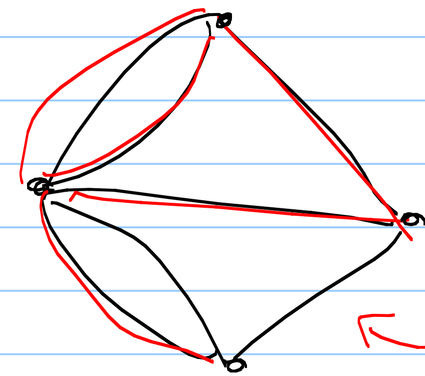
How Graphs began...

Königsberg bridge problem:

Can we walk along each bridge exactly once?

Model as a graph:

What is a walk through the city using every edge once?

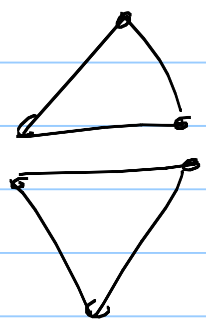


← No Euler circuit

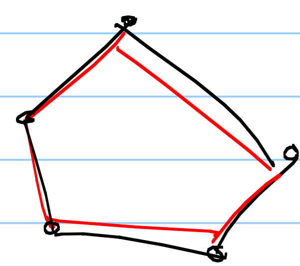
⇒ trail in graph which uses every edge.

Eulerian circuits

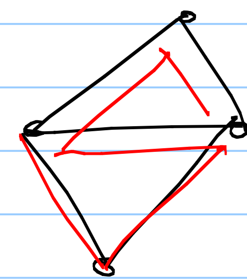
<u>Dfn</u>: An <u>Eulerian</u> circuit is a circuit which uses every edge exactly once.



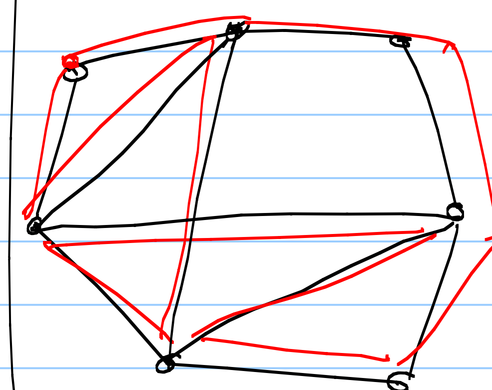
No



Yes



No

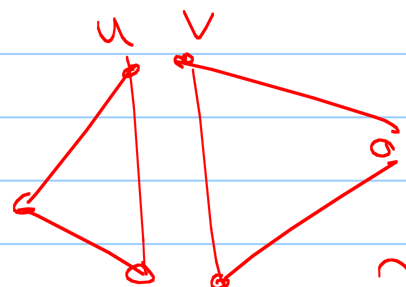


Yes

What graphs have these?

Dfn: A graph G is <u>connected</u> if for every pair of vertices u + v, there is a u-v path in G.

The <u>components</u> of G are maximally connected subgraphs.



disconnected

2 components

Thm: A graph G has an Eulerian circuit if & only if G is connected & every vertex has even degree.

Thm: Every uv-walk contains a uv-path.

pf: Induction on the length of the walk.