Math 135 - Permutations & Combinations

Note Title 4/9/2010

Announcements

- Turn in reworked midterms
- HW 7 is posted — over Ch. 5.1 - 5.4
  due next Friday in class

Last time: Pigeonhole principle

Ex: During a month with 30 days, a baseball team plays at least 1 game a day, but no more than 45 total.
Show that there is a period of consecutive days where the team plays exactly 14 games.

Solution: Let a_j = # of games played on or before the j^{th} day of the month.

$a_1, \dots, a_{30}$ ← between 1 + 45

So these are distinct, + $a_j < a_{j+1}$

Also consider: $a_1 + 14, a_2 + 14, \dots, a_{30} + 14$
↑ between 15 and 59

$$\underbrace{a_1, \ldots, a_{30}, \quad a_1+14, \ldots, a_{30}+14}$$

60 numbers, all between $\underbrace{1 \ \& \ 59}$

"objects" = 60 numbers $a_1 \ldots a_{30}+14$

"boxes" = values 1 to 59

So ≥ 2 objects wind up in same "box"

$\hookrightarrow$ $a_i = a_j + 14$

So between day j & day i, we play exactly 14 games.

Permutations (5.3)

How many ways are there to list r distinct elements from a set of size n?

Dfn: Call this $P(n,r)$.

How many ways to list n things?

$n!$

$\underline{n} \cdot \underline{(n-1)} \cdot \underline{(n-2)} \cdot \underline{\ } \cdot \underline{\ } \cdot \underline{\ } \cdot \underline{\ } \cdot \underline{\ }$

Formula: $P(n,r) = \dfrac{n!}{(n-r)!}$

$$\underbrace{\underline{n}\quad \underline{(n-1)}\quad \underline{(n-2)}\quad \underline{\cdots}\quad \underline{(n-(r-1))}}_{r}$$

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} = \frac{n(n-1)\cdots 1}{(n-r)(n-r-1)\cdots 1}$$

Ex: Suppose we have 8 runners, & will award 3 medals (gold, silver & bronze). Assuming no ties, how many different possible ways to award?

$$8 \cdot 7 \cdot 6 = \frac{8!}{5!} = P(8,3)$$

$n = 8$
$r = 3$

Ex: How many permutations of the alphabet contain the string 'ABC'?

How many perms of alphabet? $26!$

answer: $24!$

treat 'ABC' as a single letter

Combinations

order doesn't matter!

How many ways are there to choose r elements out of n?

Notation: $C(n,r) = \binom{n}{r} =$ "n choose r"

book's   everybody else   read it

Ex: How many ways to choose 2 elements from $\{1, 2, 3, 4, 5\}$?

Ans: $\{1,2\}$ $\{1,3\}$ $\{1,4\}$ $\{1,5\}$
$\{2,3\}$ $\{2,4\}$ $\{2,5\}$
$\{3,4\}$ $\{3,5\}$
$\{4,5\}$

$$\binom{5}{2} = 10$$

Thm: $P(n,r) = \binom{n}{r} \cdot P(r,r)$

pf: left side: listing r things out of n

right: choosing r things from n of them, & then ordering them $\blacksquare$

(combinatorial proof - count the same thing 2 different ways)

This gives a nice formula:

$$\binom{n}{r} = \frac{P(n,r)}{P(r,r)}$$

$$= \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!\,(n-r)!}$$

Ex: How many possible poker hands are there?
(52 different cards in deck, 5 card hands)

$$\underbrace{\binom{52}{5}}_{\text{preferred}} = \frac{52!}{5!\,47!}$$

Ex: How many bit strings of length 5 have exactly 3 ones?

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$$\binom{5}{3} = \frac{5!}{3!\,2!}$$

(Follow-up: How many bitstrings of length n have exactly r ones?)

$$\binom{n}{r}$$

Combinatorial proof:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Pf: right side: # of bit strings of length n

left side: rule of sum

every bitstring has between 0 + n ones

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \text{total number of bitstrings of length } n$$

Combinatorial Proof:

a proof which uses counting arguements to prove that both sides of an identity count the same thing

choose r things from n

eliminate n-r things from a group of n

Ex:
$$\underbrace{\binom{n}{r} = \binom{n}{n-r}}$$

ways to put r 1's in a length n bitstring

ways to have n-r 0's.

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Thm: $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$

pf: by combinatorial argument

left side: Selecting $r+1$ things from a group of $n+1$ things

right side: distinguish 1 thing from the group.

That item is either selected or not.

rule of sum
$\left\{\begin{array}{l}\text{If selected, } \binom{n}{r} \text{ ways to form the group.} \\ \text{If not selected, } \binom{n}{r+1} \text{ ways.}\end{array}\right.$

$\boxtimes$

Thm (Vandermonde's identity)

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

pf: left side: choosing r things for a group from m+n choices

right side:

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2}$$

separate into 2 groups, one of size m + one of size n

Need to choose k from one group & $r-k$ from the other in order to get r total)

$$\Rightarrow \binom{n}{k}\binom{m}{r-k}$$ (rule of product)

Then add possibilities together, since there is no overlap. (rule of sum)