Math 135 - Big O cont.

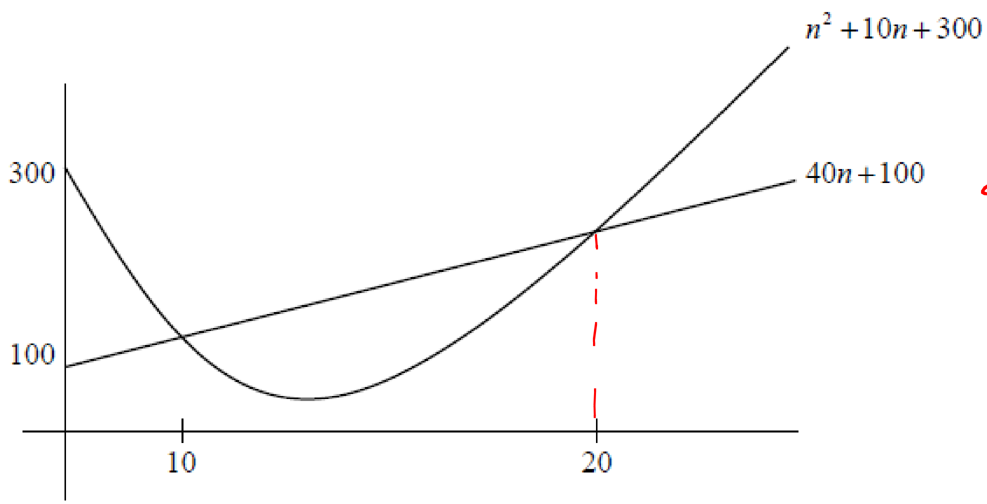
Note Title 2/19/2010

Announcements

- Midterms will come back on Monday
- Look for average so far on Friday (?)
- New HW will be out Monday

Growth of functions: Section 3.2

Consider 2 functions:



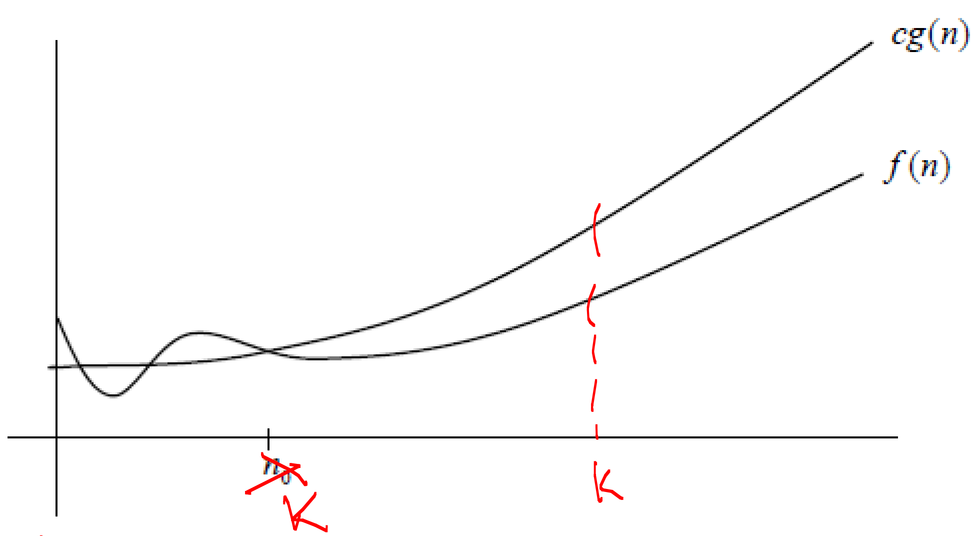
← smaller since ($\forall n \geq 20$) the function $40n + 100 \leq n^2 + 10n + 300$

Which is bigger?

Big-O:

Dfn: Let f & g be functions from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$). We say that $f(x) = O(g(x))$ if there are constants C and k such that

$$|f(x)| \le C|g(x)| \quad \text{for } x > k.$$



To prove a function $f(x)$ is $O(g(x))$:

Idea: First select a k that lets you estimate size of $f(x)$ for $x > k$.

Then look for a C that gets desired inequality.

Ex: $\underbrace{7x^2 + 3x}_{f(x)}$ is $O(\underset{\uparrow g(x)}{x^2})$.

$\underbrace{\text{If } x \geq 1,}$ $7x^2 + 3x \leq 7x^2 + 3x^2 = \underline{10}\,x^2$ $\longleftarrow$

Let $k = 1$ and $c = 10$

Then $\forall x \geq k$, $f(x) \leq c \cdot g(x)$ from above inequality, Ex: Show that $n = O(2^n)$ ←

pf: Need to show $\forall n \geq k$, $n \leq c \cdot 2^n$

When we did induction, we proved:

$\forall n \geq 1$, $n \leq 2^n$

So let $k=1$, $c=1$, & we're done.

Strategy #2: Do an induction proof!

Ex: Show that $\underbrace{\log_2 n}_{\lg} = O(n)$

pf: Use a fact: $a \le b$

$$\iff \log(a) \le \log(b)$$

Know $\forall n \ge 1$, $n \le 2^n$.

Take $\log_2$ of both sides:

$$\lg n \le \lg(2^n) = n \cdot \lg 2 = n$$

so let $k=1$, $c=1$

$\boxtimes$

Sometimes write $f(x) = O(g(x))$

Not an equality!

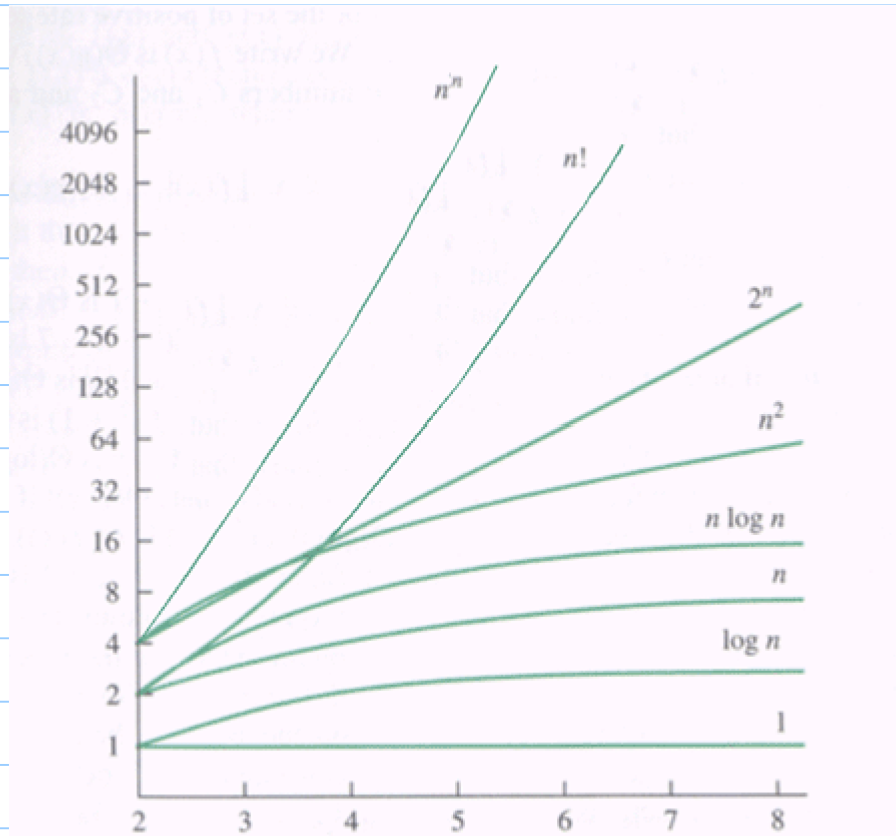
- $x^2 + 2x + 1 = O(x^2)$

also $n^2 = O(n^n)$

- $x^2 + 2x + 1 = O(x^3)$

(Really mean $f(x) \in \{\text{functions that are } O(g(x))\}$)

A big picture:



$7x^2 + 5x + 10$

Thm: Let $f(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

Then $f(x) = O(x^n)$.

pf: Consider $|f(x)| = |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0|$

triangle inequality: $|a+b| \le |a| + |b|$

So $|f(x)| \le |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0|$

$$= x^n \left(|a_n| + \frac{|a_{n-1}|}{x} + \cdots + \frac{|a_1|}{x^{n-1}} + \frac{|a_0|}{x^n} \right)$$

If $x \ge 1$, then $\le x^n \left(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| \right)$

$$|f(x)| \le x^n\left(|a_n| + |a_{n-1}| + \dots + |a_1| + |a_0|\right)$$

$$= x^n\left(\sum_{i=0}^{n} |a_i|\right)$$

Let $k = 1$

and $c = \sum_{i=0}^{n} |a_i|$ (a constant)

then $|f(x)| \le c \cdot x^n$

so $f(x)$ is $O(x^n)$

Thm: Suppose $f(x) = O(g(x))$ and $h(x) = O(p(x))$.

Then $(f+h)(x) = O(\max(g(x), p(x)))$.

Why? $f(x) + h(x) \le c_1 g(x) + c_2 p(x)$

$\le 2\max(c_1, c_2) \cdot \max(g(x), p(x))$

Give big-O estimate for $\underbrace{3n^3 + 2n^2 + n\log n + n^n}_{= O(n^n)}$

Corollary: Suppose $f_1(x)$ & $f_2(x)$ are $O(g(x))$.
Then $(f_1+f_2)(x) = O(g(x))$.

so $f_1(x) = x^3 + 3x - 12$

$f_2(x) = 5x^3 - 12x + 11$

each $O(x^3)$

$(f_1+f_2)(x) = O(x^3)$

Similarly:

<u>Thm</u>: Suppose $f(x) = O(g(x))$ & $h(x) = O(p(x))$.

Then $(f \cdot h)(x) = O(g(x)\,p(x))$

Ex: Give a big-O estimate for

$$f(x) = 3n \log(n!) + (n^2+3)\log n$$

Consider $3n \cdot \log(n!) = O(n \cdot n \log n) = O(n^2 \log n)$

$3n = O(n)$

$\log(n!) = O(n \log n)$

$(n^2+3) \log n = O(n^2 \cdot \log n)$

$n^2+3 = O(n^2)$

So $f(x) = O(n^2 \log n) + O(n^2 \log n) = \boxed{O(n^2 \log n)}$

Big-Omega

Dfn: Let f & g be functions from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$)
We say $f(x)$ is $\Omega(g(x))$ if $\exists$ positive constants C, k such that

$$|f(x)| \geq C \cdot |g(x)| \quad \text{when } x > k.$$

(Read - f is big-Omega of g).

Ex: Show $f(x) = 8x^3 + 5x^2 + 7$ is $\Omega(x^3)$.

if $x \geq 1$, $8x^3 + 5x^2 + 7 \geq 8x^3$

So let $k = 1$, $C = 8$

big-O technique $8x^3 + 5x^2 + 7 \leq 8x^3 + 5x^3 + 7x^3$

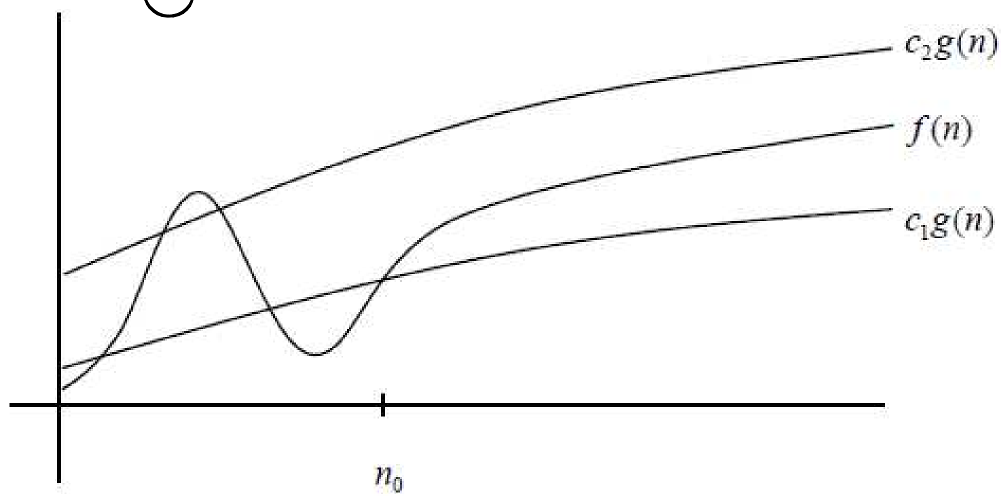
Ex: $\sum_{i=1}^{n} i = \Omega(n^2)$.

Big-Theta

Let f & g be functions ($\mathbb{R} \to \mathbb{R}$ or $\mathbb{Z} \to \mathbb{R}$).
We say $f(x)$ is $\Theta(g(x))$ if

- $f(x)$ is $\Omega(g(x))$
- $f(x)$ is $O(g(x))$

We say f & g are <u>asymptotically equivalent</u>.



Ex: $\sum_{i=1}^{n} i = \Theta(n^2).$

Why?