# Math 135: Big-O Notation (3.2)

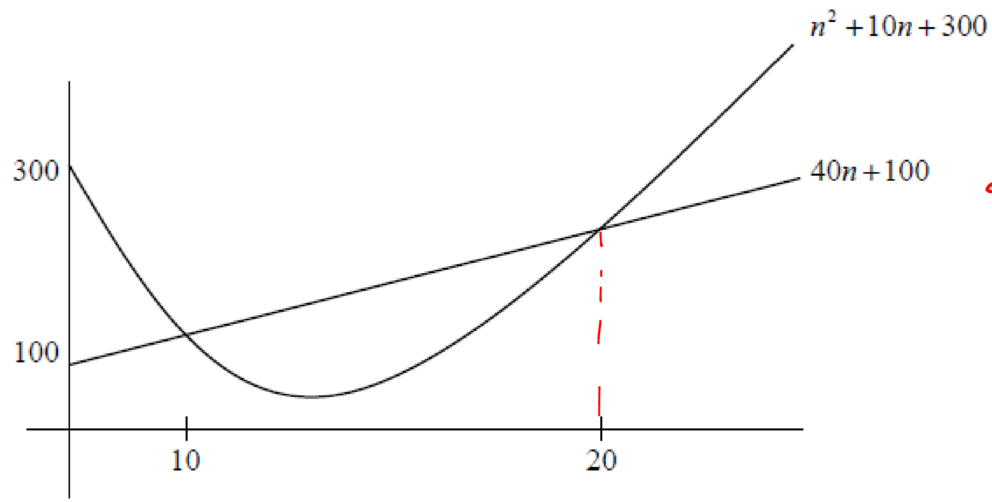
Note Title 2/12/2010

Announcements

- Practice test is up
- Midterm next Wednesday in class
- Review - Monday in class
- Solutions to HW will be handed out on Monday

# Growth of functions:

Consider 2 functions:




smaller since $\forall n \geq 20$ the function $40n + 100 \leq n^2 + 10n + 300$

Which is bigger?

# Big-O:

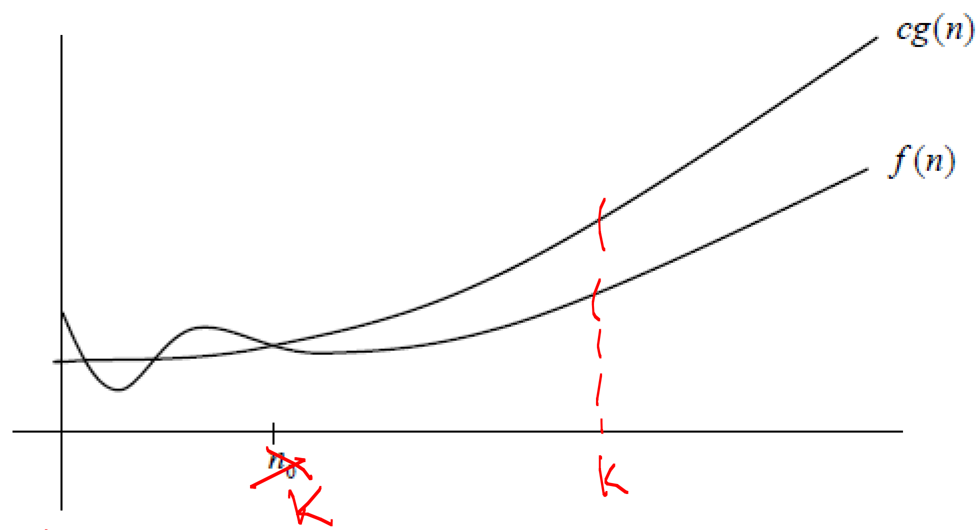
if f is big-O of g, then "$f \le g$"

**Dfn:** Let $f$ & $g$ be functions from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$). We say that $f(x) = O(g(x))$ if there are constants $C$ and $k$ such that

$$|f(x)| \le C|g(x)| \quad \text{for } x > k.$$

$\exists k, C$ s.t. $\forall x > k$, $f(x) \le c \cdot g(x)$

negate: $\forall k, \forall C\ \exists x > k$ s.t. $f(x) > c \cdot g(x)$




(here, $n_0 = k$)

Ex: $f(x) = x^2 + 2x + 1$ is $O(x^2)$

proof: Need to find $k$ & $C$.

Consider $x^2 + 2x + 1$.

If $x \geq 1$, then $x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2$

Let $k = 1$, $c = 4$

So $\forall x \geq k$, have $f(x) \leq c \cdot x^2$

$\Rightarrow f(x)$ is $O(x^2)$.

Idea: First select a $k$ that lets you estimate size of $f(x)$ for $x > k$.

Then look for a $C$ that gets desired inequality.

So can also get:

$$f(x) = x^2 + 2x + 1 \text{ is } O(x^3)$$

If $x \geq 1$, then $x^2 + 2x + 1 \leq x^3 + 2x^3 + x^3 = 4x^3$

so let $k = 1$, $C = 4$

& desired inequality holds: $f(x) \leq 4 \cdot x^3$ for $x \geq k$

Sometimes write $f(x) = O(g(x))$

Not an equality!

- $x^2 + 2x + 1 = O(x^2)$

- $x^2 + 2x + 1 = O(x^3)$

(Really mean $f(x) \in \{\text{functions that are } O(g(x))\}$)

Ex: Show that $7x^2 = \underbrace{O(x^3)}$

Spps $x \geq 1$, then $7x^2 \leq \underbrace{7x^3}$

So let $k = 1$, $C = 7$ ∎

Spps $x \geq 7$, then $7x^2 \leq \underbrace{x^3} = x \cdot x^2$

So let $k = 7$, $C = 1$. ∎

Ex: Show that $n^2$ is not $O(n)$.

pf: Harder, since we need to show no constants $C$ & $k$ can exist with $\underbrace{n^2 \le C\cdot n}$ for some $n > k$.

Observe when $n > 0$, we can divide both sides by $n$:

$$n \le C$$

Let $\underbrace{n = \max\{C, k\} + 1}$

then $n > C$ (so $n \le C$ is false)

& so $n^2$ is not $O(n)$. ∎

constant

<u>Ex</u>: $f(x) = \sin x$ is $O(1)$.

<u>pf</u>: let $k = 0$ & $C = 2$

know $\sin x \le 1 \le 2 \cdot 1$

so $\sin x = O(1)$

Ex Consider $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n$.

What is it if we want big-O?

(Two ways to do this:)

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2} \leq \frac{n^2}{2} + \frac{n^2}{2} = n^2$

   if $n \geq 1$

   So set $k = 1$, $c = 1$, get $\forall n \geq k$, $\sum_{i=1}^{n} i \leq n^2 \Rightarrow \sum_{i=1}^{n} = O(n^2)$

2. $$\sum_{i=1}^{n} i = 1+2+3+\cdots+(n-1)+n$$

$$\leq \underbrace{n+n+n+\cdots+n+n}_{n}$$

$$= n^2$$

So let $k=1$, $c=1$

$$\Rightarrow \sum_{i=1}^{n} i = O(n^2)$$

$\blacksquare$

**Ex:** Give a big-O bound for $n! = n(n-1)\cdots 1$.

$$n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1$$

$$\leq \underbrace{n\cdot n\cdot n\cdots\cdot n\cdot n}_{n} = n^n$$

$$\underbrace{\text{so let } k=1,\ c=1,\ n! \leq n^n}$$

so $n! = O(n^n)$ ←

$\log_2 8 = 3$
$\log_{10} 10000 = 4$

$\log_b a$ = power we raise b to in order to get a

**Ex:** What about $\log_2(n!)$?

pf: set $k=1$, $c=1$, + get $n! \le n^n$

take $\log_2$ of both sides:

$$\lg(n!) \le \lg(n^n) = n \lg n$$

So $\lg(n!) = O(n \lg n)$

$\boxtimes$

In induction, we showed $n \le 2^n$ for $n \ge 1$.

What big-Oh does this imply?

<u>Ex</u>: Use above to show $\log_2 n = O(n)$.

A big picture: