Math 135 - More induction (Ch 3.3)

Note Title 1/25/2010

Announcements

- HW is out, & due next Monday

Recall: Induction

A proof technique that is used to prove propositions of the form:

$$\forall n, P(n)$$

Idea:

1. Show $P(1)$ true
2. Show $\forall k > 1, \; P(k-1) \rightarrow P(k)$

Since $P(1)$ is true (by 1):

$P(1) \rightarrow P(2)$ (by 2)

$P(2) \rightarrow P(3)$ (by 2)

$P(3) \rightarrow P(4)$ (by 3)

$\vdots$

How to write inductive proofs

3 required parts

Base Case $P(1)$

Inductive Hypothesis

Assume $P(k-1)$ is true

Inductive Step

Use $P(k-1)$ to show $P(k)$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = \underline{120}$$

Thm: $\forall n \geq 4$, $2^n < n! = n \cdot \underbrace{(n-1) \cdot (n-2) \cdots 2 \cdot 1}_{= (n-1)!}$

Proof: by induction on n:

Base Case: $n = 4$: $2^4 = 16$

$4! = 24$

$16 < 24$ (so $P(4)$ is true)

Ind. Hyp: Assume $2^{n-1} < (n-1)!$ (for $n-1 > 4$)

Ind. Step: $2^n = 2 \cdot 2^{n-1} \underset{\text{by IH}}{<} 2 \cdot (n-1)!$

$2(n-1)! < n \cdot (n-1)! = n!$ since $2 < n$

Suppose n friends have a water balloon fight. Each moves to a location (so that all distances between friends are distinct).
Next, each throws their balloon at the closest target.

<u>Claim:</u> If n is odd, then at least one person stays dry.

<u>proof</u>: by induction on n = # of people

<u>Base case</u>: <u>$n=1$</u>: He stays dry.

<u>$n=3$</u>:  2 closest must throw at each other, so 3rd survives.

dry

IH: Assume true for smaller odd #s.
If $n-2$ people have a water balloon fight, 1 stays dry.
(next smallest odd #)

IS: Suppose we have n people, + n is odd.
Consider the 2 people realizing smallest distance. They throw at each other.
Remove them.
Now have $n-2$ people, so by IH know someone stays dry, call him Bob.
Re-add our 2 nearest neighbors.
Bob is still dry, since nearest neighbors don't throw at him
$\Rightarrow$ 1 person stays dry. $\square$

The Gossip Problem

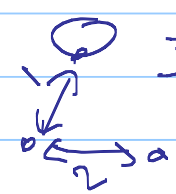
- There are n people, each of whom knows one secret.
- Every time 2 people call each other, they tell each other <u>all</u> the secrets they know.

How many phone calls are needed for everyone to know <u>all</u> of the secrets?

$n=2$

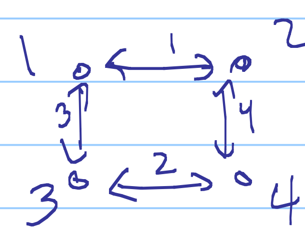
1 call

$n=3$



3 calls

$n=4$



4 calls

call 1: 1 knows $\{1,2\}$, 2 knows $\{1,2\}$

call 2: 3 & 4 know $\{3,4\}$

call 3

Claim: If $n \geq 4$, then $2n-4$ suffice.

Pf:

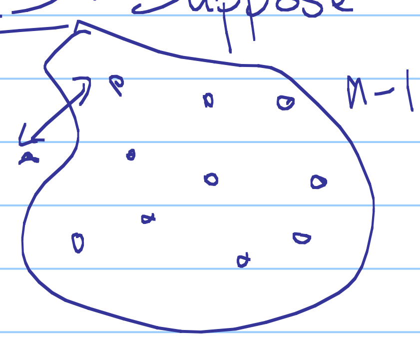
Base case: p_1, p_2, p_3, p_4

have p_1 call p_2, then p_3 call p_4, then p_1 call p_3, then p_2 call p_4

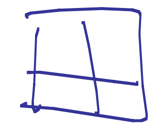
4 calls, & $2 \cdot 4 - 4 = 4$ ✓

IH: For $n-1$ people, $\underbrace{2(n-1)-4}_{= 2n-6}$ calls suffice.

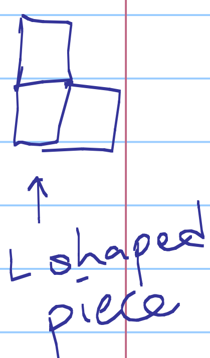
IS: Suppose we have n people.

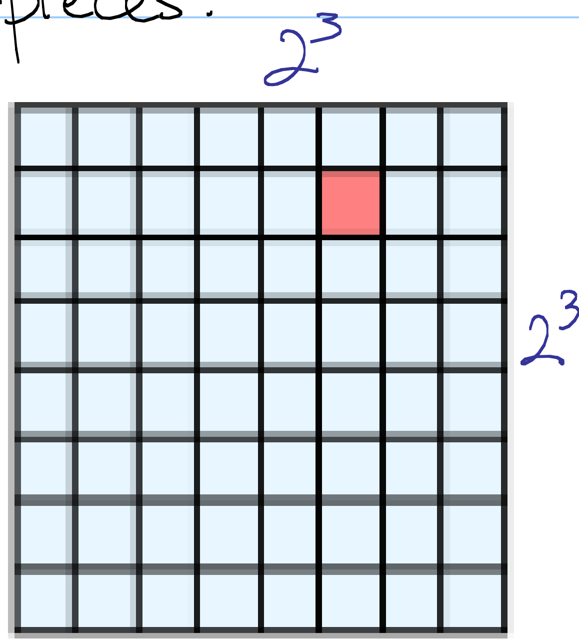


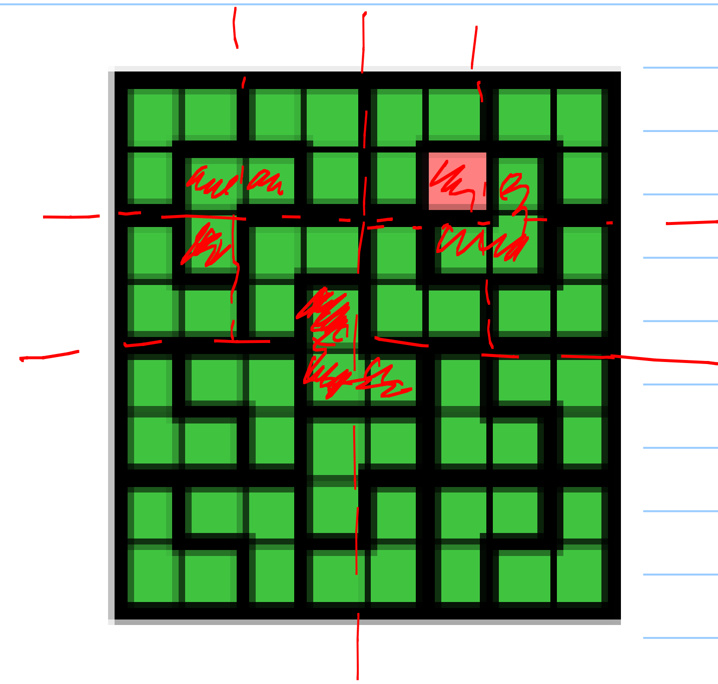
have person n call person $n-1$.
Now apply IH, & have people $1 \ldots n-1$ share all secrets in $2n-6$ calls.
Then have person $n-1$ call person n back. total: $2n-4$ calls ∎



Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled with L shaped pieces.







Use induction on n:

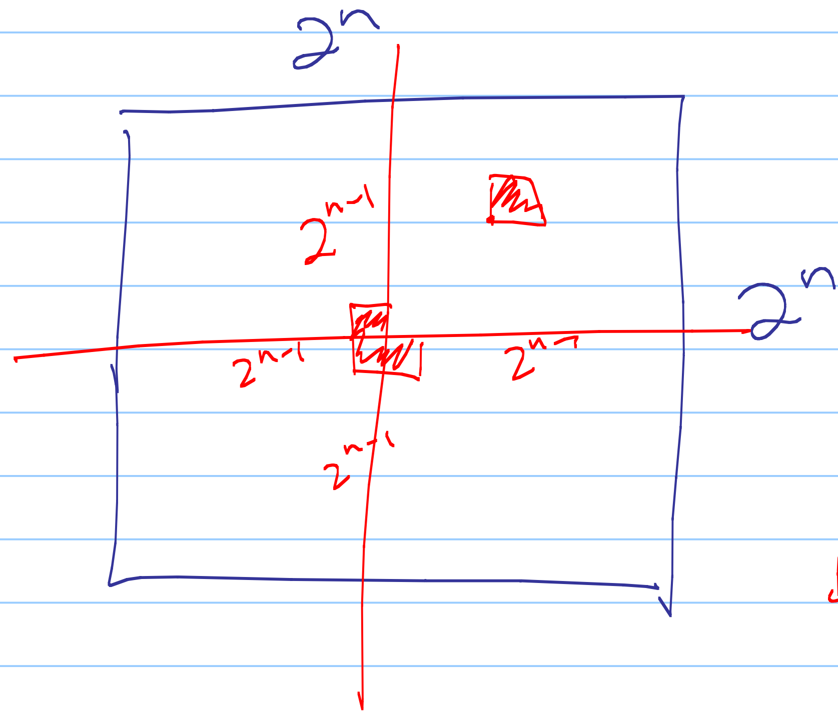
<u>Base case:</u> $n=1$

$2^1 \times 2^1$ board

Remove any 1 box, & L covers the rest.

<u>IH:</u> Assume we can tile any $2^{n-1} \times 2^{n-1}$ chessboard with 1 tile removed using L-shaped pieces.

IS: Consider $2^n \times 2^n$ board w/ 1 square removed.



Put L tile over 3 quadrants with no tiles already removed.

Now have 4 $2^{n-1} \times 2^{n-1}$ squares with 1 tile in each removed.
By IH, can tile each with L-shaped tiles.

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