Math 135 - More fun with Sets

Note Title 2/1/2010

Announcements

- HW due tomorrow
- Look for next HW to be out on Wed./Thursday
- No office hours next Thusday (2/11) (sorry, doctor appointment)

Sets: some more definitions

Let S be a set. If S has exactly n (unique) elements, then we say S is finite, with cardinality n, written $|S| = n$.

S is said to be infinite if it is not finite.

What are infinite sets?

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Dfns (cont)

The <u>power set</u> of S, $P(S)$ or 2^S, is the set of all subsets of S.

<u>Ex</u>: Let $S = \{0, 1, 2\}$. What is the power set of S?

$$2^S = \{ \{0\}, \{0,2\}, \{1\}, \{0,1\}, \{1,2\}, \emptyset, \{2\}, \{0,1,2\} \}$$

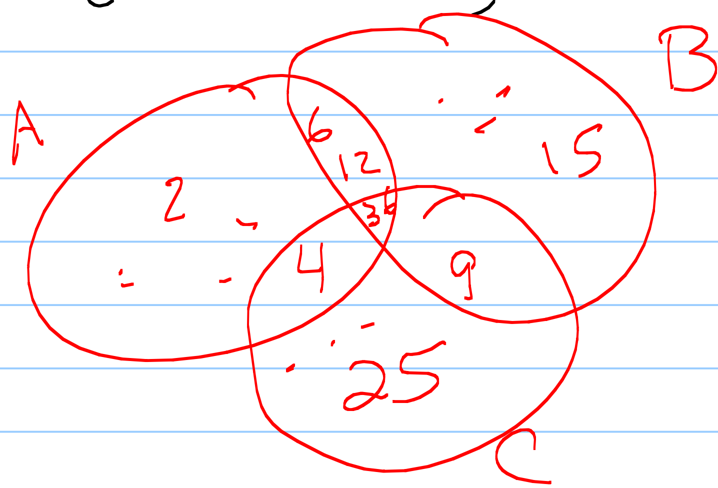
<u>Ex</u>: What is the power set of $\emptyset$?

$$2^{\emptyset} = \{\emptyset\} = \{ \{\} \}$$

Venn Diagrams

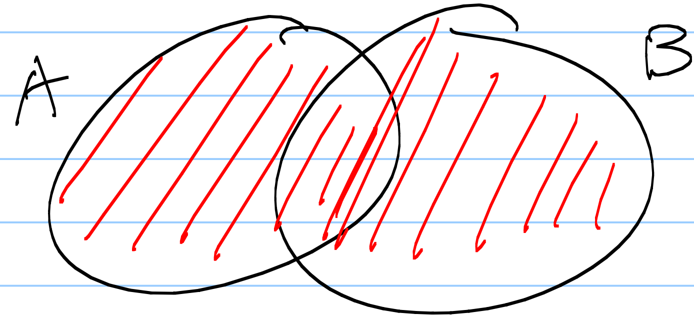
Sometimes we want a picture of how sets interact.

Ex: $A = \{n \in \mathbb{N} : n \text{ is even}\}$
$B = \{n \in \mathbb{N} : n \text{ is divisible by } 3\}$
$C = \{n^2 : n \in \mathbb{N}\}$

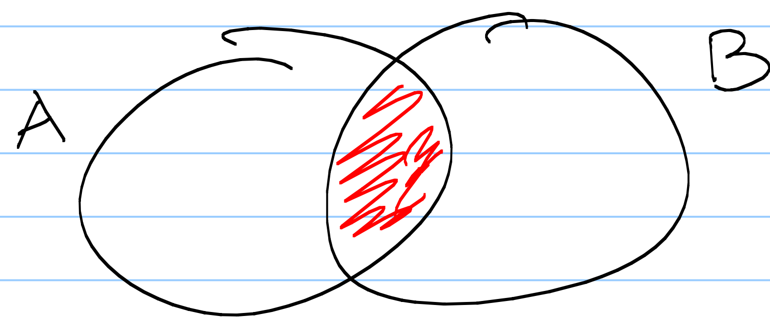


More Definitions

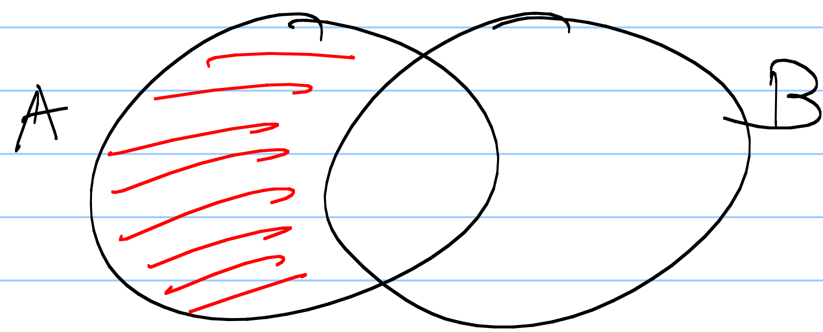
Union : $A \cup B = \{ x \mid x \in A \overset{\text{or}}{\vee} x \in B \}$



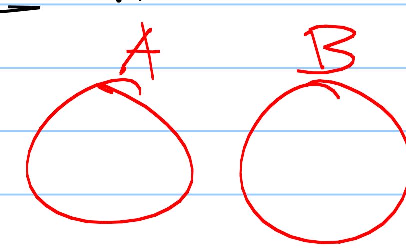
Intersection : $A \cap B = \{ x \mid x \in A \underset{\text{and}}{\wedge} x \in B \}$



Set Difference: $A - B = \{x \mid x \in A \text{ and } x \notin B\}$



Dfn: Two sets are called **disjoint** if their intersection is empty, i.e. $A \cap B = \emptyset$.



Examples

$A = \{2, 7, \{a,b\}, \pi\}$

$B = \{\sqrt{2}, \pi, a, b\}$

$C = \{\{a\}, b, \{a,b\}\}$

$A \cap C = \{\{a,b\}\}$

$A \cup B = \{2, 7, \{a,b\}, \pi, \sqrt{2}, a, b\}$

$A \cap B = \{\pi\}$

$(A \cap C) \cup B = \{\sqrt{2}, \pi, a, b, \{a,b\}\}$

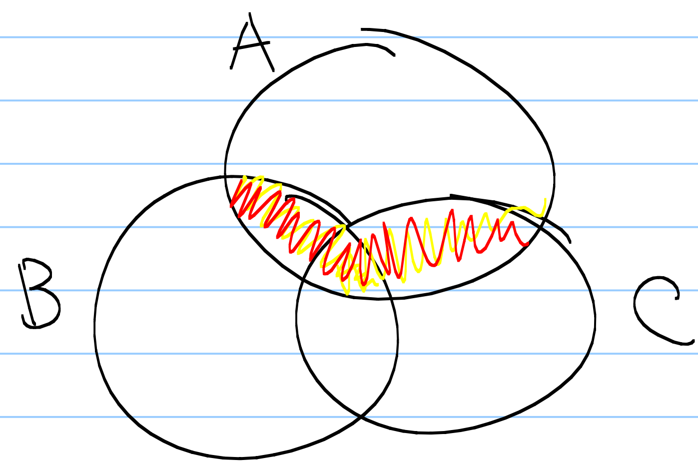
$B - C = \{\sqrt{2}, a, \pi\}$

Set identities

<u>Thm</u>: For all sets A, B & C,

$$\underbrace{A \cap (B \cup C)} = \underbrace{(A \cap B) \cup (A \cap C)}$$

(so $\cap$ distributes over $\cup$)



a Venn diagram is <u>not</u> a proof

Proof: Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

1. $\underline{A \cap (B \cup C)} \subseteq \underline{(A \cap B) \cup (A \cap C)}$

Let $x \in A \cap (B \cup C)$.

$\Rightarrow x \in A$ and $\underbrace{x \in B \cup C}_{\substack{\parallel \\ x \in B \text{ or } x \in C}}$

If $x \in A$ and $x \in B$, then $x \in A \cap B$

If $x \in A$ and $x \in C$, then $x \in A \cap C$.

One of these must hold, since $x \in A$ and $x \in B \cup C$.
So $x \in A \cap B$ or $x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$

② Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

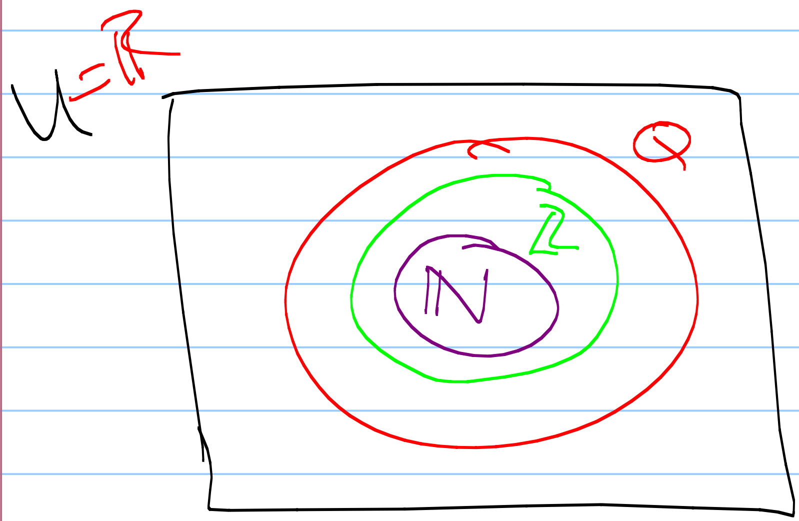
Let $x \in (A \cap B) \cup (A \cap C)$.

So $x \in A \cap B$ or $x \in A \cap C$.

- If $x \in A \cap B$, then $x \in A$ and $x \in B$.
  Now $x \in B \Rightarrow x \in B \cup C$
  So $x \in A$ and $x \in B \cup C$
  $\Rightarrow x \in A \cap (B \cup C)$
- If $x \in A \cap C$, then $x \in A$ and $x \in C$.
  Now $x \in C \Rightarrow x \in B \cup C$.
  So $x \in A$ and $x \in B \cup C$
  $\Rightarrow x \in A \cap (B \cup C)$ ∎

The Universe

Many times, all of the sets we are interested in come from a single large set called the <u>universe</u>.



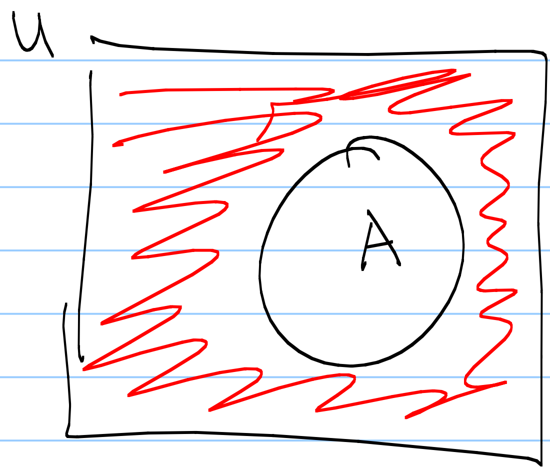
$U = \mathbb{R}$

$\left.\begin{array}{l} \mathbb{Z} \\ \mathbb{Q} \\ \mathbb{N} \end{array}\right\}$ $\begin{array}{l} \mathbb{N} \subsetneq \mathbb{Z} \\ \frac{1}{2} \in \mathbb{Q} \\ \mathbb{Z} \subsetneq \mathbb{Q} \end{array}$

$\sqrt{2} \notin \mathbb{Q} \Rightarrow \mathbb{Q} \subset \mathbb{R}$

Complementation: Relative to U, the complement of A is

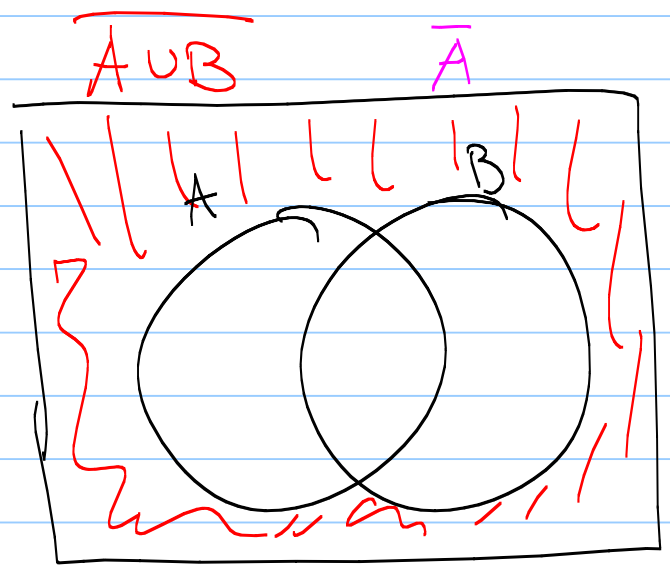
$$\overline{A} = U - A = \{x \in U : x \notin A\}$$

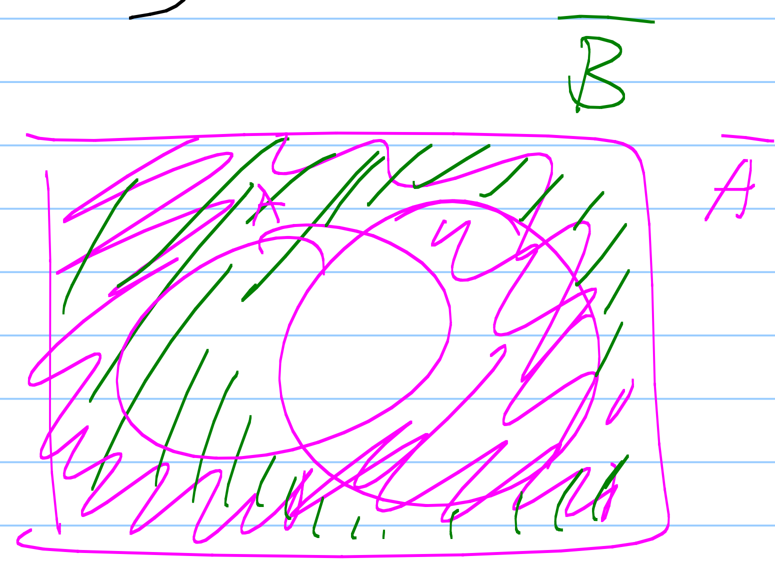


De Morgan's Laws

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

← (look familiar?)





Prove that $\overline{A \cap B} = \bar{A} \cup \overline{B}$. ✓

pf: How do we show two sets are equal?

(i) Show $\overline{A \cap B} \subseteq \bar{A} \cup \overline{B}$

take $x \in \overline{A \cap B}$

So $x \in U$ and $\underbrace{x \notin A \cap B}$,

so $x \in U$ and $[x \notin A$ or $x \notin B.]$

If $x \notin A$ and $x \in U$, then $x \in \bar{A}$.

If $x \notin B$ and $x \in U$, then $x \in \bar{B}$.

So $x \in \bar{A}$ or $x \in \bar{B}$.

$\Longrightarrow x \in \bar{A} \cup \bar{B}$

② Show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$

then $x \in \overline{A}$ or $x \in \overline{B}$

If $x \in \overline{A}$, then $x \in U$ and $x \notin A$.