Math 135 - Functions

Note Title 2/5/2010

Announcemts

- Midterm 1: Wed, Feb. 17 ~ in class
- HW3 is up — due next Friday

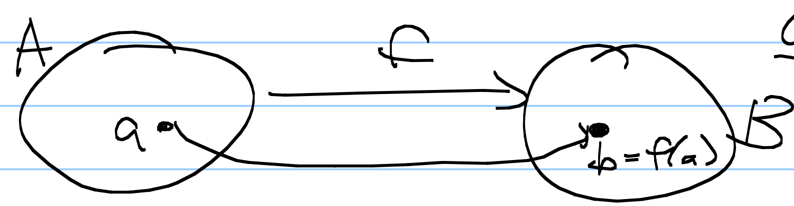
Functions

Let A & B be sets. A function from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ where $a \in A$, $b \in B$.

Often write $f: A \to B$ to denote a function f.

A is the domain of f, & B is the co-domain.



Examples

① $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x + 1$

② Truth table $t: \{T, F\}^2 \longrightarrow \{T, F\}$

Domain: $\{T, F\} \times \{T, F\}$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Codomain: $\{T, F\}$

$t(TT) = T$

③ $f:\{1,2,3,4,5\} \to \mathbb{N}$

$f(x) = \left\lceil \frac{x}{2} \right\rceil$ ← ceiling function

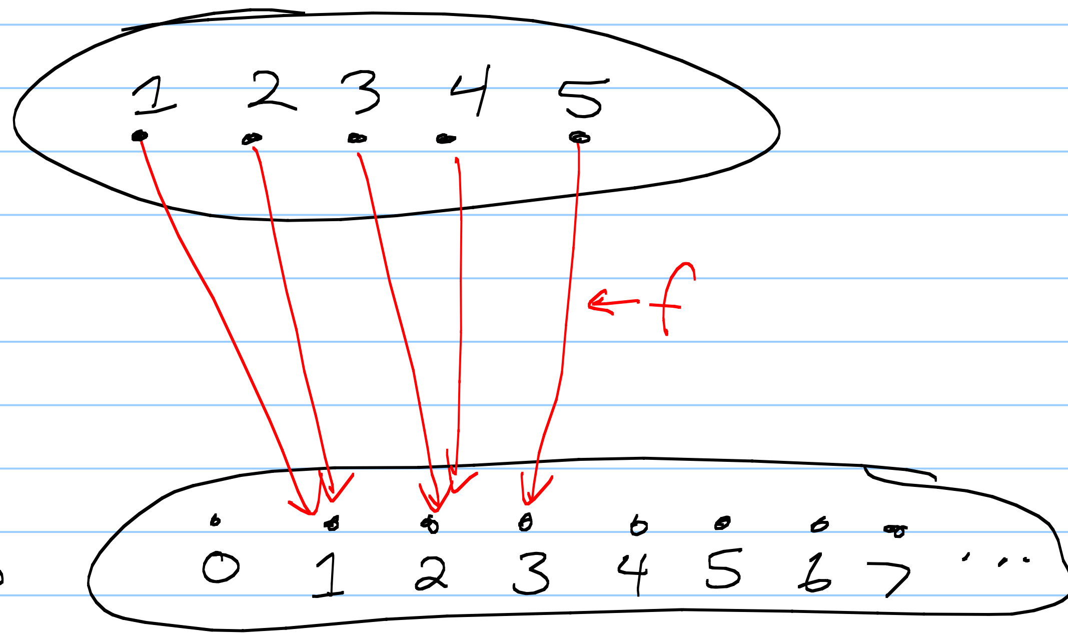
$f(1) = \left\lceil \frac{1}{2} \right\rceil = 1$ domain

$f(2) = \left\lceil \frac{2}{2} \right\rceil = 1$

$f(3) = \left\lceil \frac{3}{2} \right\rceil = 2$

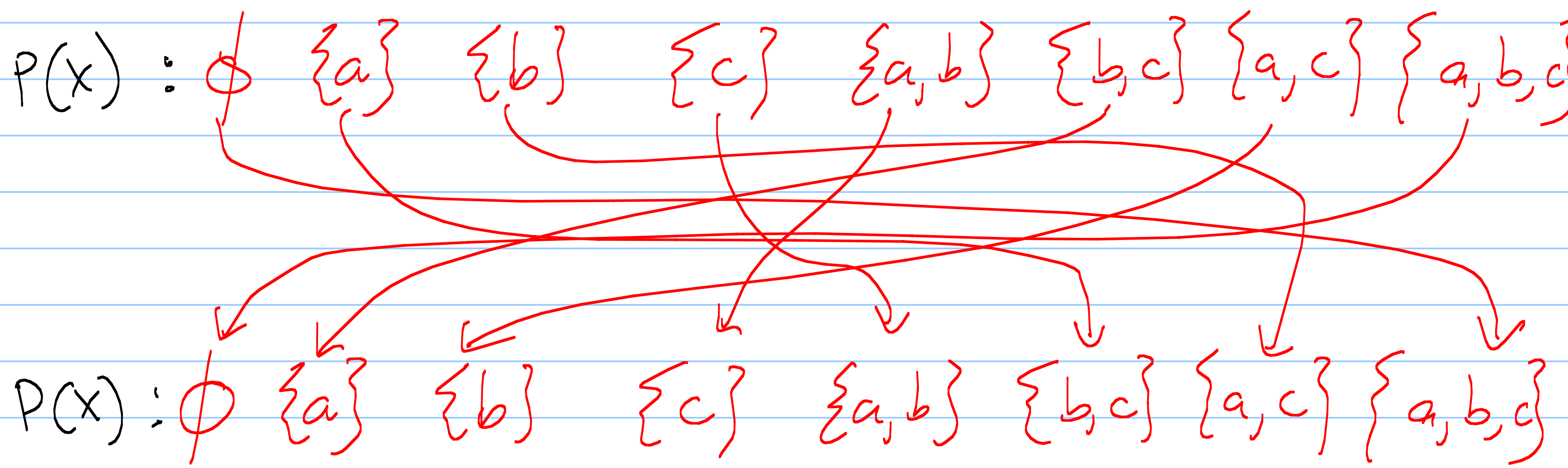
$f(4) = 2$

$f(5) = 3$ codomain



Ex: Let $X = \{a, b, c\}$ and $c : P(X) \to P(X)$ be the function:

$$c(A) = X - A$$



Dfn: A function $f: A \to B$ is <u>one-to-one (1-1), or injective,</u> if & only if $f(a)=f(b)$ implies $a=b$.

Such a function is said to be an <u>injection</u>.

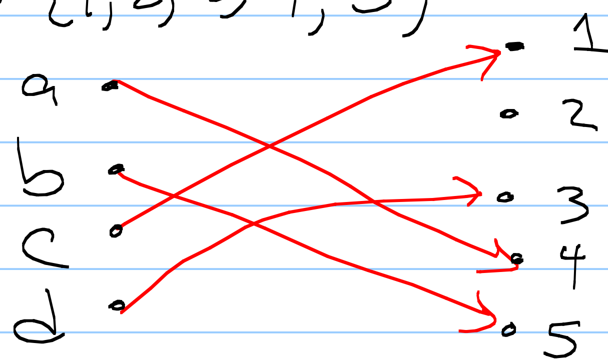
logic notation:

$$\forall a \, \forall b \, [f(a)=f(b) \to a=b]$$

So for these functions, no element in B has more than one element of A mapping to it.

<u>Ex:</u> $f : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$

$f(a) = 4$
$f(b) = 5$
$f(c) = 1$
$f(d) = 3$



yes, this is 1-1

<u>Ex:</u> $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

No, because:

$x = 1$ then $f(x) = 1$

$x = -1$ then $f(x) = 1$

Prove that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x+1$, is injective.

Pf: Need to show that $f(x) = f(y) \to x = y$.

Suppose $f(x) = f(y)$.

So $x+1 = y+1$

Now subtract 1 from both sides & they are still equal

$\Rightarrow x = y$

$\boxtimes$

$b \in B$

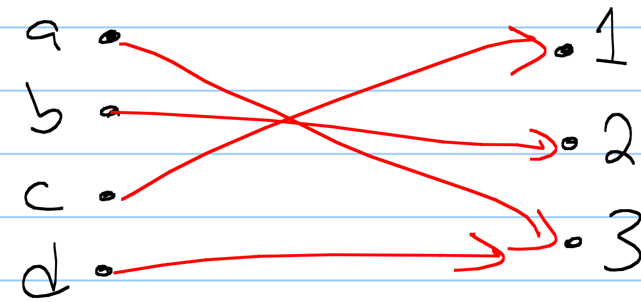
Dfn: A function is called onto (or surjective) if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

In logic: $\underbrace{\forall b \; \exists a \;\; f(a) = b}$,

So for these functions, every element of B must be an "output" of f.

Examples (1) $f: \{a, b, c, d\} \to \{1, 2, 3\}$

$f(a) = 3$
$f(b) = 2$
$f(c) = 1$
$f(d) = 3$



Onto? yes
1-1? No — 3 got hit twice

(2) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

Is it onto? No - ~~for y = -1~~,

there is no x s.t. $f(x) = -1$

(3) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x+1$

onto?

Yes. Consider $b \in \mathbb{Z}$,

$f(b-1) = b$

Dfn: A function is a bijection if it is both 1-1 and onto.

Ex: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x+1$

(already showed onto and 1-1)

Dfn: The **identity function** on A, $i_A : A \to A$, is the function $i_A(a) = a \quad \forall a \in A$.

Ex: $i_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$

$$i_{\mathbb{N}}(n) = n$$

$$A = \{1, 2, 3\}$$

$$i_A(1) = 1$$
$$i_A(2) = 2$$
$$i_A(3) = 3$$

Dfn: Suppose f is a bijection. The *inverse* of f, written f^{-1}, is the function

$$f^{-1}: B \to A \quad \text{where} \quad f^{-1}(b) = a \iff f(a) = b.$$

Ex: What is the inverse of $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x+1$?

$$f^{-1}(y) = y-1$$

Ex: $f: \mathbb{Q} \to \mathbb{Q}$, $f(x) = \frac{x}{2} + 3$

Is it 1-1? Need $f(a) = f(b) \Rightarrow a = b$

Yes!

① $\frac{a}{2} + 3 = \frac{b}{2} + 3 \Rightarrow a = b$

② Spps $a \neq b$. Show $f(a) \neq f(b)$

Is it onto? Give you $\ell \in \mathbb{Q}$

$2(\ell - 3)$ will map to ℓ, & is in $\mathbb{Q}$.

Inverse: $y = \frac{x}{2} + 3 \Rightarrow y - 3 = \frac{x}{2}$

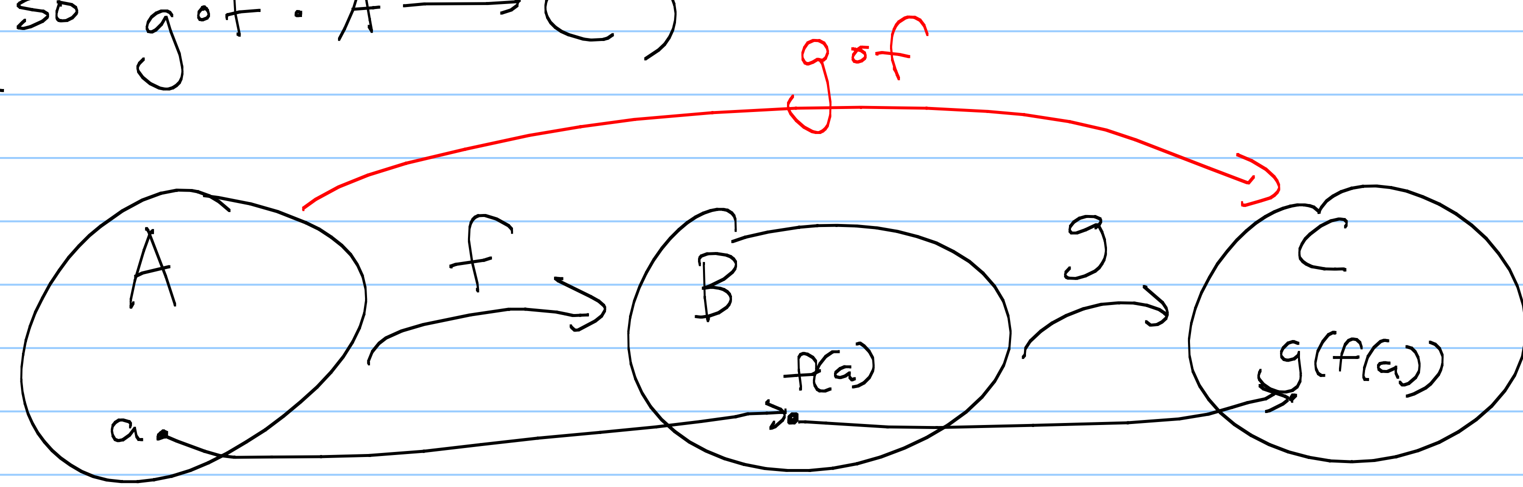
$\Rightarrow 2(y-3) = x$ so $f^{-1}(a) = 2(a-3)$

Composition of functions

Given $f: A \to B$ and $g: B \to C$, the composition of f and g, written $g \circ f$, is the function

$$(g \circ f)(a) = g(f(a))$$

(so $g \circ f : A \longrightarrow C$)



Ex: Let $f: \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 2x+3$
and $g: \mathbb{Z} \to \mathbb{Z}$ with $g(x) = 3x+2$.

What is $g \circ f$? $(g \circ f)(x) = g(f(x))$

$= g(2x+3) = 3(2x+3) + 2$

$= 6x + 11$

$f^{-1}(f(x)) = x$

<u>Thm</u>: Functions $f: A \to B$ and $g: B \to A$ are inverses of each other ~~if~~ if and only if $f \circ g = i_B$ and $g \circ f = i_A$.

<u>Proof</u>: Next time...