Math 135 - Counting

Note Title 3/26/2010

Announcements

- Turn in HW
- Midterm 2 next Wednesday
- Review session on Monday
  - Will repost cheat sheet (Master thm is worded differently than your version)

$$T(k) = aT\left(\frac{k}{b}\right) + f(k)$$

set $k = \frac{n}{b}$

Final word on recurrences

Recursion tree:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$T\left(\frac{n}{b}\right) = aT\left(\frac{n}{b^2}\right) + f\left(\frac{n}{b}\right)$$

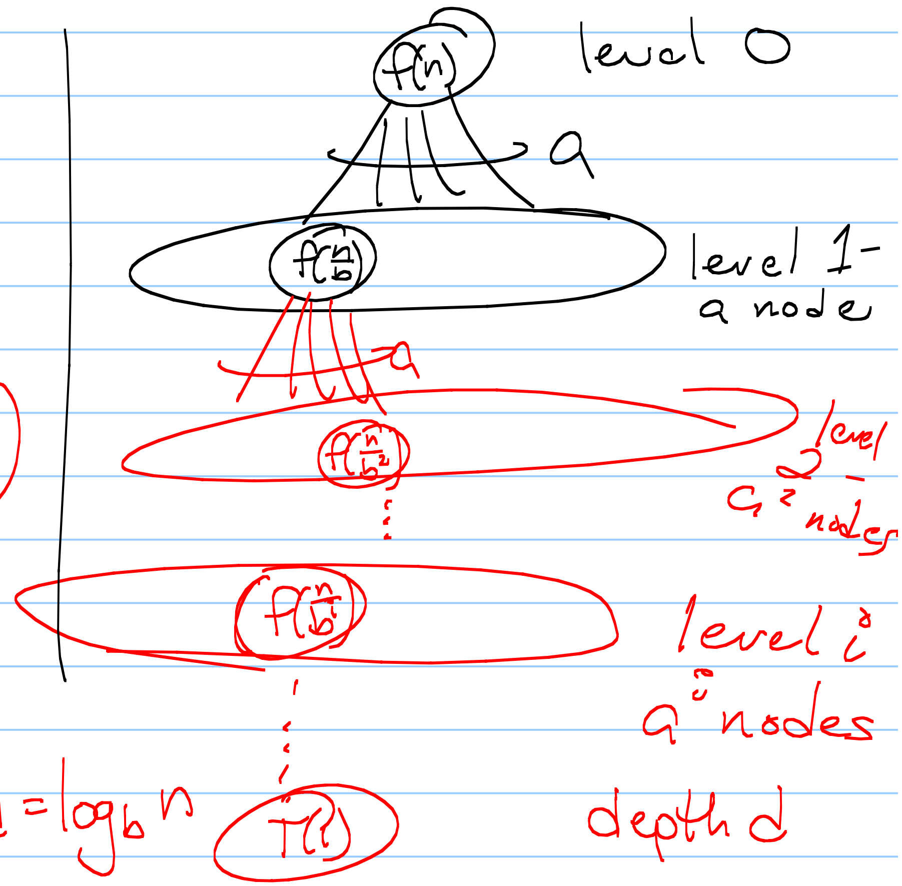
$$T\left(\frac{n}{b^2}\right) = aT\left(\frac{n}{b^3}\right) + f\left(\frac{n}{b^2}\right)$$

$\vdots$

$$T\left(\frac{n}{b^i}\right)$$

$\vdots$

$$T(1) \Rightarrow \frac{n}{b^d} = 1 \Rightarrow d = \log_b n$$



$$\underbrace{T(n) = \sum_{i=0}^{\log_b n}}_{\text{depth } d} \underbrace{a^i}_{\#\text{ nodes}} \underbrace{f\left(\frac{n}{b^i}\right)}_{\text{work per node}} = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \dots + \underbrace{a^{\log_b n}}_{= n^{\log_b a}} \cancel{f(1)} \to 1$$

Master thm just recognizes that this is increasing or decreasing geometric series.

Ex: $T(n) = 2T\left(\frac{n}{2}\right) + n$

$$\sum_{i=0}^{\log_2 n} 2^i \left(\frac{n}{2^i}\right) = \underbrace{n + 2 \cdot \frac{n}{2} + 2^2 \cdot \frac{n}{2^2} + \dots + n^{\log_2 2} \cdot 1}_{\text{these are all equal, \& there are } \log_2 n \text{ of them}}$$

$$= n \log_2 n$$

$\underline{\text{Ex}}$: If $f(n) = n^k$, have

$$T(n) = \sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^k = \sum_{i=0}^{\log_b n} a^i \cdot \frac{n^k}{(b^i)^k} = n^k \cdot \sum_{i=0}^{\log_b n} \left(\frac{a}{b^k}\right)^i$$

If $a < b^k$: then $\frac{a}{b^k} < 1$

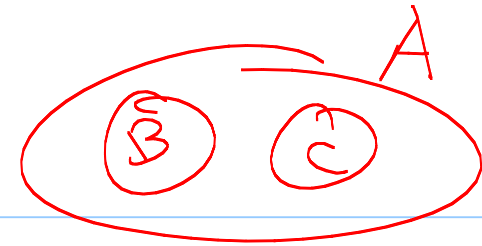
then $T(n) \leq n^k \sum_{i=0}^{\infty} \left(\frac{a}{b^k}\right)^i = n^k \underbrace{\left(\frac{1}{1-\frac{a}{b^k}}\right)}_{\text{constant}}$

$= O(n^k)$

Counting - Ch 5

2 Basic Principles

1. Rule of Sum
2. Rule of Product



Rule of Sum

If B and C are disjoint and $A = B \cup C$, then $|A| = |B| + |C|$.

We can split A into non-overlapping subsets, so we can sum the sizes of B & C.

Ex: Need a math representative to a committee. There are 37 students available & 12 faculty.

Total possible choices $= 37 + 12 = 49$

Ex: $A = \{(x,y) \in \{1,2,\dots,n\}^2 : x = 4 \text{ or } x = 5\}$

(Recall: $\{1,2,\dots,n\}^2$ is set of ordered pairs (x,y) where $1 \le x \le n$ and $1 \le y \le n$.)

Ex: $\{1,2,3\}^2 = \{(1,1), (1,2), (1,3), (2,1), \dots\}$

$$A = \underbrace{\{(x,y) : x = 4\}}_{n} \cup \underbrace{\{(x,y) : x = 5\}}_{n}$$

$$|A| = n + n = 2n$$

② Rule of product:

Suppose a set can be formulated as a sequence of k choices.
Then, if there are n_1 ways to make first choice, n_2 to make second, etc.,

$$|A| = n_1 \cdot n_2 \cdots n_k$$

Ex: How many binary strings of length n?

$$\underline{2}\cdot\underline{2}\cdot\underline{2}\ \cdots\ \underline{\ }\ \underline{\ }\ \cdots\ \underline{2}$$

n spots $\qquad\qquad\qquad$ 0 or 1

Total # of binary strings of length n

$$= \underbrace{2\cdot 2\cdot \cdots \cdot 2}_{n} = 2^n$$

Ex: Chairs in an auditorium will be labeled with a letter and a positive integer ≤ 100.

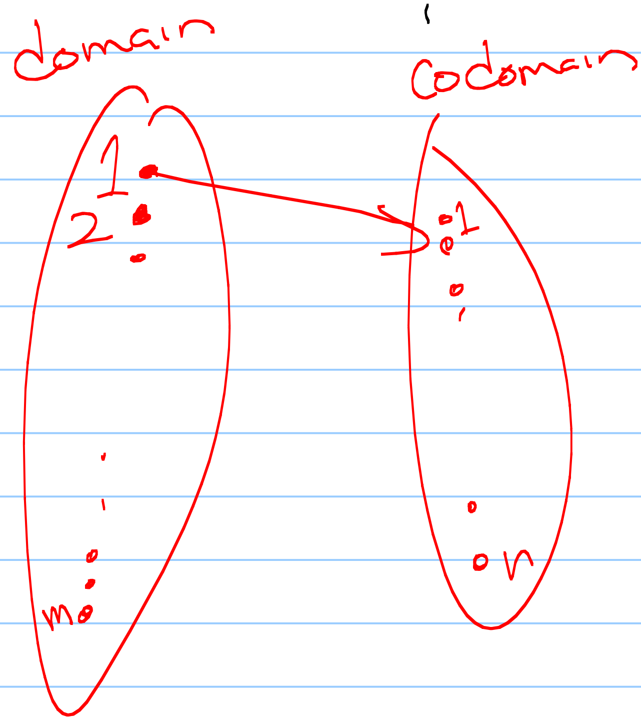
How many chairs are possible?

$$\le 26 \cdot 100 = 2{,}600$$

26 letters

100 possible #s

Ex: How many different functions from a set with m elements to a set with n elements?

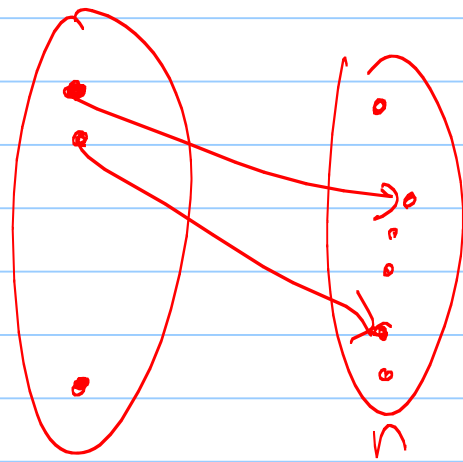


n	choices for	1
n	"	2
n	"	3
	$\vdots$	
n		m

$$\text{total} = \underbrace{n \cdot n \cdot \ldots \cdot n}_{m} = n^m$$

Ex: How many one-to-one functions from a set w/ m elements to a set w/ n elements?

each element in domain points to unique element in co-domain



n

m

if $m > n$, are no 1-1 functions

if $m = n$, $m! = m(m-1)(m-2)\cdots$ possible functions

if $m < n$, $n(n-1)(n-2)\cdots(n-(m-1))$

More complicated

In one version of the programming language BASIC, variables could be 1 or 2 alphanumeric characters.

- Had to begin with a letter.
- 5 reserved keywords were forbidden
- No distinguishing lower + upper case.

How many variables were possible?

$$\# \text{vars} = (\# \text{ 1 char variables}) + (\# \text{ of 2 char vars})$$

$$= 26 + 26 \cdot (26 + 10) - 5$$

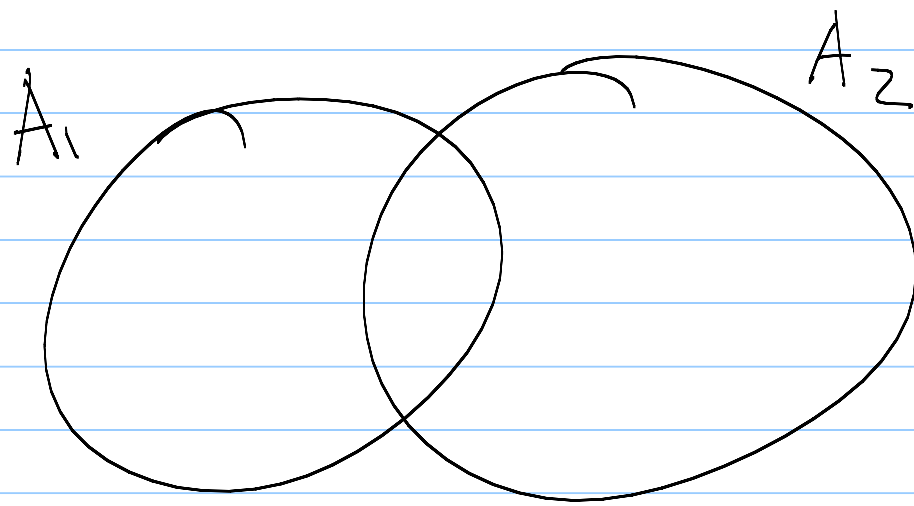
Ex: Suppose you need to make a password.

- 6 to 8 characters long
- Uppercase letters or numbers
- At least 1 digit.

How many are possible?

Principle of Inclusion/Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



Ex: How many bit strings of length n either start with a 1 or end with 00?