Math 135: Discrete Mathematics, Spring 2010
Homework 9

Due *in class* on May 3, 2010

For this homework, you may write up solutions with 1 partner; both of you will receive the same grade based on your joint writeup.

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1. Let G be a simple graph on n vertices. Show that G is a tree if and only if G is connected and has $n-1$ edges.

2. (a) Let G be a graph with n vertices and m edges. Prove that the average degree of the vertices in G is $2m/n$.
   (b) Use part (a) to prove that every planar graph has a vertex of degree at most 5.

3. Give a combinatorial proof of the following identity: If n and k are natural numbers with $0 \leq k \leq n$, then

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$$

4. Show that an edge in a graph is a cut edge if and only if this edge is not part of any cycle in the graph.

5. Prove or disprove the following:
   (a) A graph is connected if and only if some vertex is connected to every other vertex.
   (b) Every edge in a tree is a cut edge.

6. Extra Credit problem: A *tournament* is a directed complete graph (so for every pair of vertices u and v, exactly one of uv and vu is an edge of the graph).
   (a) How many different tournaments are there with n vertices?
   (b) Show that every tournament has a Hamiltonian path.