Math 135: Discrete Mathematics, Spring 2010
Homework 6

Due *in class* on Friday, March 26, 2010

For this homework, you may write up solutions with 1 partner; both of you will receive the same grade based on your joint writeup.

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1. Let f_n be the n^{th} Fibonacci number, defined as $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$ and $f_1 = 1$. (Hint: Remember, induction is your friend when doing recurrences!)

   (a) Prove that $\sum_{i=1}^{n}(f_i)^2 = f_n f_{n+1}$ whenever n is a positive integer.

   (b) Show that $f_{n+1}f_{n-1} - (f_n)^2 = (-1)^n$ when n is a positive integer.

2. Suppose that you find on your Math 135 instructor's desk a toy with 3 wooden pegs, one of which has 8 disks of different sizes neatly stacked (largest on the bottom, smallest on the top) on it. A card near the toy tells you the rules of playing:

   - Your goal is to move all of the disks from the first peg to one of the other two pegs.
   - Only one disk may be moved at a time.
   - No disk may ever be placed on a smaller disk.

   Let R_n be the minimum number of steps required to move n disks on a peg of such a toy to another peg. Impress your instructor by giving (and justifying) a recurrence for R_n and the solving it.

3. Give *exact* solutions to the following recurrences. Show your work.

   (a) $A(n) = -4A(n-1) + 5A(n-2)$, $A(0) = 2$, $A(1) = 8$.

   (b) Find the solution to the same recurrence as part (a), with $A(0) = 2$, $A(1) = 4$.

   (c) $C(n) = 2C(n-1) + n + 5$, $C(0) = 0$.

4. Given *general form* solutions to the following recurrences. (Note: this means you don't have to solve for the constants!)

   (a) $A(n) = 7A(n-1) - 16A(n-2) + 12A(n-3) + n4^n$

   (b) $B(n) = 4B(n-1) - 4B(n-2) + (n^2+1)2^n$

   (c) $C(n) = 7C(n-2) + 6C(n-3)$

5. Solve the following *asymptotically*. You need to argue why your bounds are correct (either by induction or using summations and recursion trees or using Master theorem). However, you do not need to prove that your big-O bounds are formally correct; for example, if you get a bound of $n/2 \log n + 12$, you may just say that it is $O(n \log n)$.

   For each, assume small constant base cases, such as $A(1) = O(1)$ or $A(2) = O(1)$.

   (a) $A(n) = 5A(n/5) + n^3$

   (b) $B(n) = 4B(n/4) + \sqrt{n}$

   (c) $C(n) = 9C(n/2) + (3n^2 + 2)$

   (d) $D(n) = 3D(n/3) + 28n$

   (e) $E(n) = 2E(\sqrt{n}) + 1$

   (f) $F(n) = F(n/2) + F(n/4) + F(n/6) + F(n/12) + n$