The Zipper Foldings of the Diamond

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Introduction

A zipper folding of a polygon P given a source point $x \in \partial P$ is the polyhedron generated by identifying all points in ∂P equidistant from x, measured along the perimeter of P, in essence "zipping" the boundary of the polygon. A theorem of Alexandrov shows that as long as every glued point has nonnegative curvature, then any zipper folding of a convex polygon leads to unique convex polyhedron (where a doubly covered polygon is considered a "flat" polyhedron). Alexandrov's theorem is existential, but a more recent constructive proof by Bobenko and Izmestiev allows for the explicit construction of the polyhedron by solving a certain differential equation [2]. An implementation of the constructive algorithm has been coded by Stefan Sechelmann[1], which will output the folded convex polyhedron given a input triangulation of the polygon with gluing instructions. However, it is difficult to extract the creases and adjacencies from the initial polygon in their final output polyhedron. We seek a more combinatorial approach to computing this information. Previous work has also looked at determining all the combinatorially different polyhedra obtained via foldings, primarily for regular convex polygons as well as a few other shapes such as the Latin cross [3].

In this paper, we classify and compute the convex foldings of a diamond shape which are obtained via zipper foldings. Our primary goal was to seek a simpler combinatorial approach to testing for the correct set of folds, or crease pattern.

As was observed by Alexandrov and noted in [1], there are a finite number of possible crease patterns. However, in our experience, verifying or discounting a crease pattern has been surprisingly difficult in more complex polygons, since checking a crease pattern either involves seeing if a paper model will fold (highly prone to error) or attempting to compute the folding in a program such as Mathematica (which leads to numerical issues, among other things).

Our primary result in this work is simply that there are 21 combinatorially distinct convex polyhedra. (Note that there are 7 polyhedra which have non-triangular faces and 4 "flat" polyhedra all of which occur at isolated points where the crease pattern changes). The polyhedra shown in Figure 1 represent the 10 polyhedra with triangular faces, and the solid dots represent the 11 isolated polyhedra noted above. Together, these represent the zipper foldings of the diamond.

Enumerating the Foldings

Denote the 4 vertices of our polygon as A, B, C, D and the source and end points (where the zipping ends) are denoted by S and E respectively, and let the edges of the diamond be unit length. We suppose S is contained on the edge AB and let $0 \leq \epsilon \leq 1$ be the distance from A to S. Because of the diamond's symmetry, we only need to examine the crease patterns for S on the edge AB to determine all foldings. In our foldings, all polyhedra will have at most 6 vertices, resulting from gluing A, B, C, and D to some other point on the perimeter, as well as the vertices S and E, which each glue to themselves. We are interested in the actual adjacencies in the final folded polyhedron; this network of edges forms an adjacency graph, often called the graph of the polyhedron.

Our techniques for computing these foldings break down into several relevant categories. The first (and simplest) are the flat foldings when the entire polygon folds into a doubly-covered polygon. For example, when $\epsilon = 0$, the vertices B and D zip together and result in the shape forming a flat doubly covered regular triangle; flat foldings also occur here when $\epsilon = .5, .75$, and 1. The next simplest cases are those in which the final polyhedron has one or more faces

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[1] http://www3.math.tu-berlin.de/geometrie/ps/software.shtml#AlexandrovPolyhedron

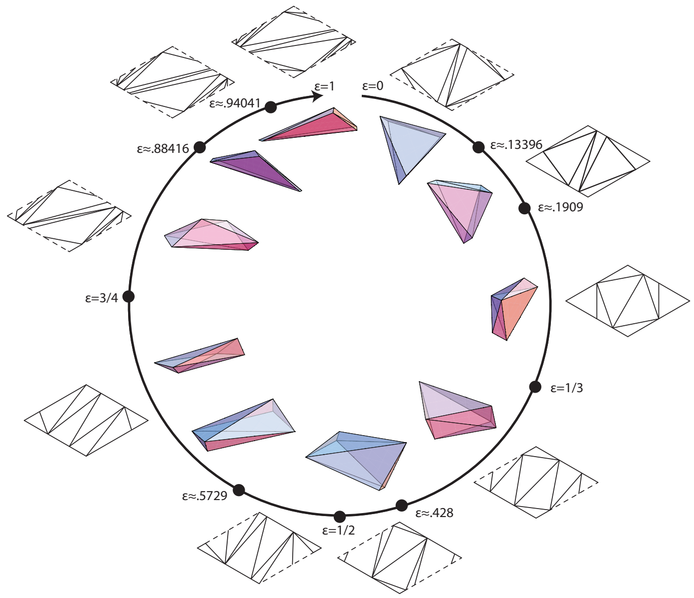

Figure 1: All the crease patterns for the zipper foldings of the diamond taken from sample values between each transition point. Dashed lines indicate that a crease extends over an edge.

that are not triangular. These cases occur for isolated values of ϵ within the interval $[0, 1]$, shown as vertices along our circle in Figure 1.

Within the subintervals of $[0, 1]$ bounded by the special values of ϵ described above, the resulting polyhedra vary continuously without a change in the adjacency graph, yielding 10 combinatorially different foldings which "flip" at isolated values where the polyhedron either folds flat or when two triangular faces become coplanar. The computation of the actual polyhedron is handled based on whether the graph of the polyhedron is 4-regular or not; if not, in our shape it will always consist of vertices of degree 3,4, and 5. When the graph has at least one vertex with degree 3, then the polyhedron is substantially easier to realize in $\mathbb{R}^3$, computationally speaking. In this shape, this results from the fact that when we have such a graph, we can decompose the final polyhedron into three tetrahedra. Our solution arose by considering the adjacency pattern and identifying these tetrahedra, then locating the points relative to each other and determining if the polyhedron resulting from their identification in $\mathbb{R}^3$ was convex.

In folding patterns where all vertices have degree 4, realization of the vertices in $\mathbb{R}^3$ is not as simple as degree 3 case. In [3], the authors describe a method for constructing an octahedron by splitting it into 2 smaller hexahedra who share an edge that is an internal diagonal of the octahedron. They vary the length of this edge until the dihedral angles of the faces incident to the edge match. We utilize a different method that also reduces a partial polyhedron to one parameter of change. Computationally speaking, these patterns are more difficult because a single crease is split into different segments inside the polygon. In order to compute these foldings, we altered the original polygon to be *non-convex* and verified the crease pattern in this related polygon.

References

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