In class exercises: structural induction

1. Define a collection $\mathcal{S}$ of binary strings recursively as follows:

   - $0 \in \mathcal{S}$ and $1 \in \mathcal{S}$.
   - If $w \in \mathcal{S}$, then $0w0, \qquad 0w1, \qquad 1w0, \qquad 1w1$ all belong to $\mathcal{S}$.

   Prove by structural induction that every string in $\mathcal{S}$ has odd length.

2. Define the collection $\mathcal{P}$ of properly parenthesized strings recursively as follows:

   - The empty string ϵ belongs to $\mathcal{P}$.
   - If $x, y \in \mathcal{P}$, then $(x)y \in \mathcal{P}$.

   Prove by structural induction that every string in $\mathcal{P}$ contains the same number of left parentheses as right parentheses.

3. A *rooted binary tree* is defined recursively as follows:

   - A single vertex is a rooted binary tree.
   - If T is a rooted binary tree, then a new root may be attached above the root of T.
   - If T_1 and T_2 are rooted binary trees, then a new root may be attached above the roots of T_1 and T_2.

   Prove by structural induction that the vertices of every rooted binary tree can be colored black and white so that adjacent vertices receive different colors.