

Transition: DS + Algorithms

More
Recurrences



Defn A linear homogeneous recurrence has the form

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_d a_{n-d}$$

where $C_1, \dots, C_d$ are constant and $C_d \neq 0$

The order of the recurrence is d .

Examples:

$$a_n = 3a_{n-1} + 2a_{n-2} \rightarrow \text{order 2}$$

$$T(n) = 0 \cdot T(n-1) + 0 \cdot T(n-2) + 0 \cdot T(n-3) + 4 \cdot T(n-4) \quad T(n) = 4T(n-4) \quad \text{order 4}$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{order 2}$$

Examples: Yes or no? Is it linear hom rec?

$$P(n) = 1.06 P(n-1)$$

yes $\rightarrow$ order 1

$$H(n) = 2H(n-1) + 1$$

no $\rightarrow$ added fn

$$R(n) = R(n-1) + (R(n-2))^2 \rightarrow \text{no, not linear}$$

$$a_n = \sqrt{3} a_{n-5}$$

yes $c_5 = \sqrt{3}$
order 5 c_1, c_2, c_3, c_4
all = 0

$$f_n = n \cdot f_{n-1}$$

n is not constant

$$M(n) = 2M\left(\frac{n}{2}\right) + n$$

no

$\uparrow$ term
 $n/2$ is not $n-d$

How to solve:

Generally, these types have solutions which are exponential.

So $\rightarrow$ look for equations of the

$$\text{form } a_n = r^n$$

where r is some constant

If a solution of this form exists,

$$\text{have: } a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_d a_{n-d}$$

$$\Downarrow$$
$$r^n = C_1 r^{n-1} + C_2 r^{n-2} + \dots + C_d r^{n-d}$$

meth!

divide common term:

$$r^{n-d}$$

$$\Downarrow$$
$$r^d = C_1 r^{d-1} + C_2 r^{d-2} + \dots + C_d$$

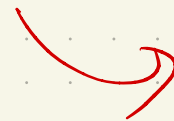
So: The sequence $a_n = r^n$ is a solution



r is a solution of the characteristic eqn:

$$\underline{r^d} - c_1 r^{d-1} - \dots - c_d = 0$$

How to find r ? math!



Worksheet:

$$a_n = 3a_{n-1} + 10a_{n-2}$$

Ex: $B(n) = 2B(n-1) + B(n-2)$

$$r^2 = 2r + 1$$

$$r^2 - 2r - 1 = 0$$

$$(r-1)^2 = 0$$

... This is a bit different!

Finding general solutions

- If r is a non-repeated root of the char. eqn, then r^n is a solution to the recurrence.
- If r is a repeated root with multiplicity k then $r, n \cdot r^n, \dots, n^{k-1} r^n$ are all solutions
- Any linear combination will satisfy the recurrence, but need to use initial values to find constants

Huh?

Examples

- $P(n) = 1.06 P(n-1)$

$$P(0) = 10,000$$

$$x - 1.06 = 0 \Rightarrow \text{root is } x = 1.06$$

so $P(n) = C \cdot (1.06)^n$

what is C ?

$$\bullet A(n) = A(n-1) + 2A(n-2)$$

$$\bullet A(0) =$$

$$A(1) =$$

$$\Rightarrow x^2 - x - 2 = 0 \text{ had roots}$$

$$x = 2 \text{ and } x = -1$$

$$\text{So: } A(n) = C_1 \cdot 2^n + C_2 \cdot (-1)^n$$

- Suppose a characteristic equation for $C(n)$ simplifies to:

$$(x-2)^3 (x-5)^2 (x+9) = 0$$

Then roots are:

$$\text{and } C(n) \geq$$

So: ~~final~~ take away!

We can find runtimes, as long as
the recurrence is linear &
homogeneous!

Q: What if it's not??

Two others we'll study here,
since they are useful in computing.

- linear inhomogeneous

- divide & conquer

Linear inhomogeneous recurrences

These are just linear ones, with an added function $g(n)$ at end:

$$f(n) = c_1 f(n-1) + \dots + c_d f(n-d) + g(n)$$

linear homogeneous

inhom. part

Examples:

$$F(n) = F(n-1) + F(n-2) + 1$$

linear hom

inhom

$$A(n) = 4A(n-1) - 2^n + n^2$$

linear hom

inhom.

Next time: How to solve? if $g(n)$ is "simple" $\rightarrow$

Method to solve

① "Ignore" $g(n)$, & find the general solution for the linear homogeneous part. How?

↳ e.g. $n!$

② Find a general solution for $g(n)$
↑ tricky
↳ "particular solution"

③ Add them together

④ Use base cases to solve for constants.
(usually we'll just do big-O)

So: Step 2 is the critical
unknown.

Depending on $g(n)$, this can be
hard.

But: If $g(n) = \underbrace{[\text{polynomial}]}_{\text{lin hom}} \times \underbrace{S^n}_{\text{some constant} + g(n)}$

then can do it.

Ex: $g(n) = (n^2 + 1) \cdot 5^n$ poly (deg 2)
or $g(n) = (5n - 2) \cdot 1^n$ $s = 5$
 $s = 1$
poly

can't solve:

$$g(n) = 5^n \cdot 2^{5n}$$

$\neq \underbrace{(\text{poly}) \cdot 5^n}_{g(n) = 1 \text{ or } = 1 \cdot 1^n}$

poly of deg 0 $= 1 \cdot n^0 \cdot 1^n$

ln $\leftrightarrow$ hom

$$a_n = 6a_{n-1} - 9a_{n-2}$$

$$r^2 - 6r + 9 = 0$$

6 ± 0

$(r-3)^2$

$$a_n = C_1 \cdot 3^n + C_2 \cdot n \cdot 3^n$$

$2^{\sqrt{n}}$
 2^{2^n}

Process

Is s a char root?

No

try a general
solution of the
form:
 $s^n \cdot (\text{poly of deg } k)$

new
constants!

k is deg of $g(s)$
 $z^n \cdot (\text{poly of deg } 0) = z^n \cdot \text{const}$

yes

What is its
multiplicity?
Let this
be m

Try a general solution
of the form

$n^m \cdot (\text{poly. of deg } k) \cdot s^n$

Example

$$\cancel{f(0) = 1}$$

$$f(n) = 4f(n-1) + 3^n$$

deg = 1

Step 1

char eqn

$$f(n) = 4f(n-1)$$

$$r = 4$$

$$(r - 4) \neq 0$$

$$s = 3$$

Step 2:

$$g(n) = 3^n = 1 \cdot (3^n)$$

poly of deg ≥ 0
 $1 \cdot n^0$
 $3^n \cdot C_2$

Step 3:

$$a_n = C_1 \cdot 4^n + C_2 \cdot 3^n$$

Step 4:

$$f(n) = 4f(n-1) + \underbrace{2n^2}_{\text{deg } 2} \cdot \underbrace{3^n}_{s=3}$$

$$\left(\frac{n^2}{2} + 5\right) 3^n$$

Step 1: same!

$$C_1 \cdot 4^n$$

$$\begin{aligned} 2n^2 \cdot n^3 \\ = 2n^5 \\ \text{deg } 5 \end{aligned}$$

Step 2: $g(n) = n^2 \cdot 3^n$
 poly of $\text{deg } 2$ $\nearrow$ $s=3$
 $k=2$

try: $(C_2 \cdot n^2 + C_3 n + C_4)$
 poly of $\text{deg } 2$

Step 3: $a_n = C_1 \cdot 4^n + (C_2 n^2 + C_3 n + C_4) 3^n$

Another: $a_n = 3a_{n-1} + 4a_{n-2} + n \cdot 4^n$

Step 1: linear hom part

$$a_n = 3a_{n-1} + 4a_{n-2}$$

$$\hookrightarrow r^2 - 3r - 4 = 0$$

$$(r-4)^2 (r+1) = 0$$

$$C_1 \cdot 4^n + C_2 (-1)^n$$

$$C_1 \cdot 4^n + C_2 n \cdot 4^n + C_3 (-1)^n$$

Step: $n \cdot 4^n$ $s=4$

$\deg \hookrightarrow$ poly

4

is a root, with mult = 1

try $n^1 (C_3 \cdot n + C_4) \cdot 4^n$

$$n^2 (C_4 n + C_5) \cdot 4^n$$

In end: add Step 1 & Step 2

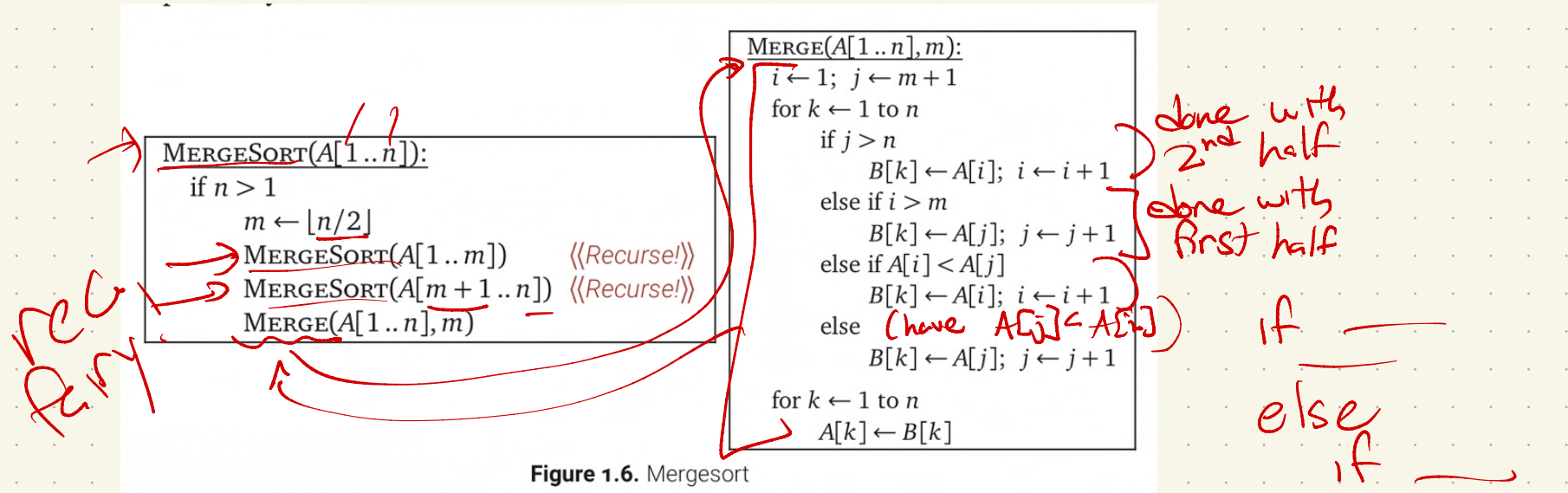
$$a_n = C_1 \cdot 4^n + C_2(-1)^n + n(C_3 n + C_4) \cdot 4^n$$

$$= C_1 4^n + C_2(-1)^n + C_3 n^2 4^n$$

$$+ C_4 n \cdot 4^n$$
$$= O(n^2 4^n)$$

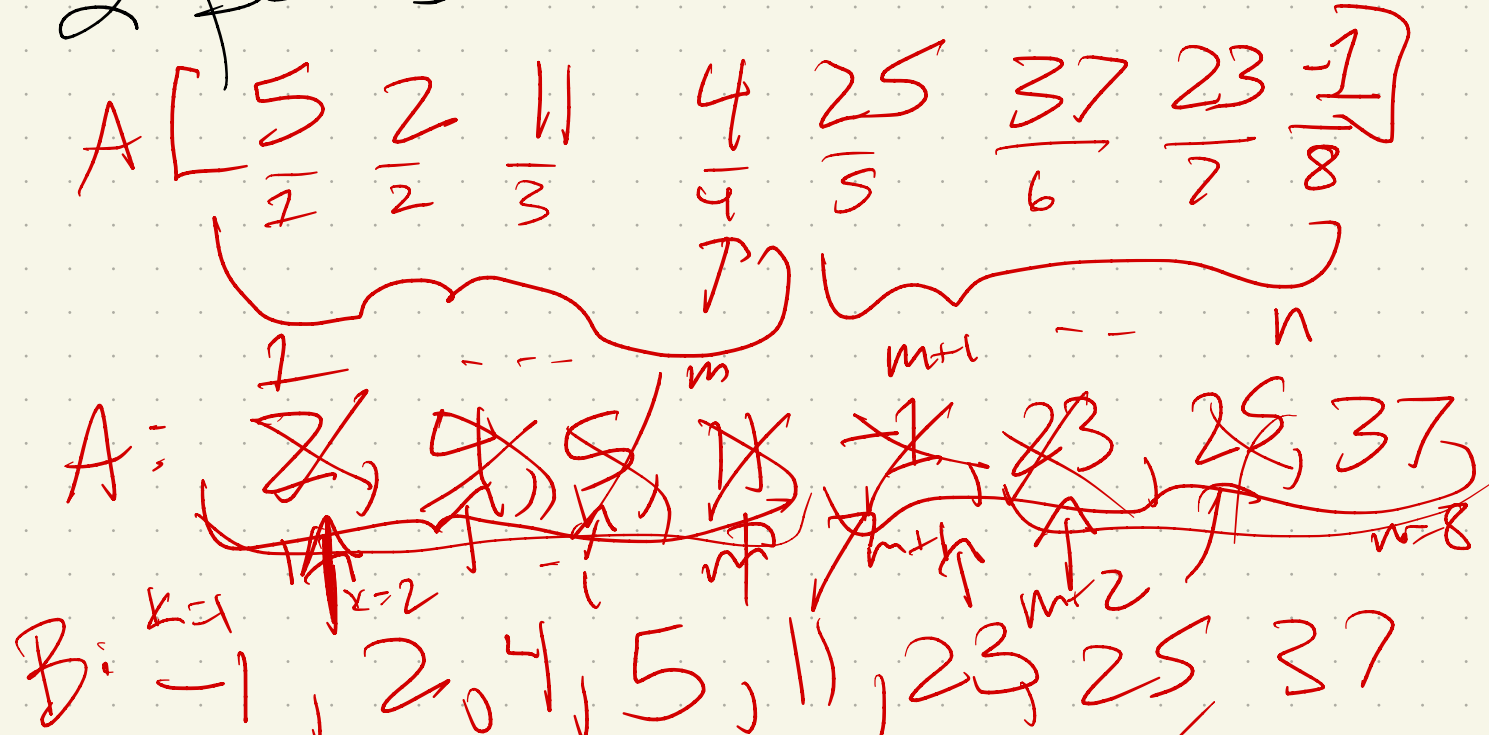
$$a_n = C_1 \cdot 4^n + C_2 n 4^n + C_3(-1)^n + n^2(C_5 n + C_6) \cdot 4^n$$

Next reading: Divide & conquer



Mergesort: 2 parts

$n = 8$
 $m = n/2 = 4$



Part 1: Merge works

Setup: Given $A[1..n]$ and an index m
with $1 \leq m \leq n$ where $A[1..m]$ +
 $A[m+1..n]$ are sorted, MERGE
correctly sorts $A[1..n]$ by end.

How?

Part 2:

Induction of length of $A[1..n]$:

Read & see how
it goes!

Runtime:

MERGESORT(A[1..n]):

if $n > 1$

$m \leftarrow \lfloor n/2 \rfloor$

$\rightarrow$ MERGESORT(A[1..m]) $\llcorner$ Recurse! $\llcorner$

$\rightarrow$ MERGESORT(A[m+1..n]) $\llcorner$ Recurse! $\llcorner$

MERGE(A[1..n], m)

$\hookrightarrow O(n)$

MERGE(A[1..n], m):

$i \leftarrow 1; j \leftarrow m+1$

for $k \leftarrow 1$ to n

if $j > n$

$B[k] \leftarrow A[i]; i \leftarrow i+1$

else if $i > m$

$B[k] \leftarrow A[j]; j \leftarrow j+1$

else if $A[i] < A[j]$

$B[k] \leftarrow A[i]; i \leftarrow i+1$

else

$B[k] \leftarrow A[j]; j \leftarrow j+1$

for $k \leftarrow 1$ to n

$A[k] \leftarrow B[k]$

$\sum_{k=1}^n 5 = 5n$
 $= O(n)$
 $\sum_{k=1}^n 1 = n$
 $O(n) = \sum_{k=1}^n 1$

Figure 1.6. Mergesort

Let $T(n)$ = runtime of MS on
input of size n

$$T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lfloor \frac{n}{2} \rfloor) + O(n)$$

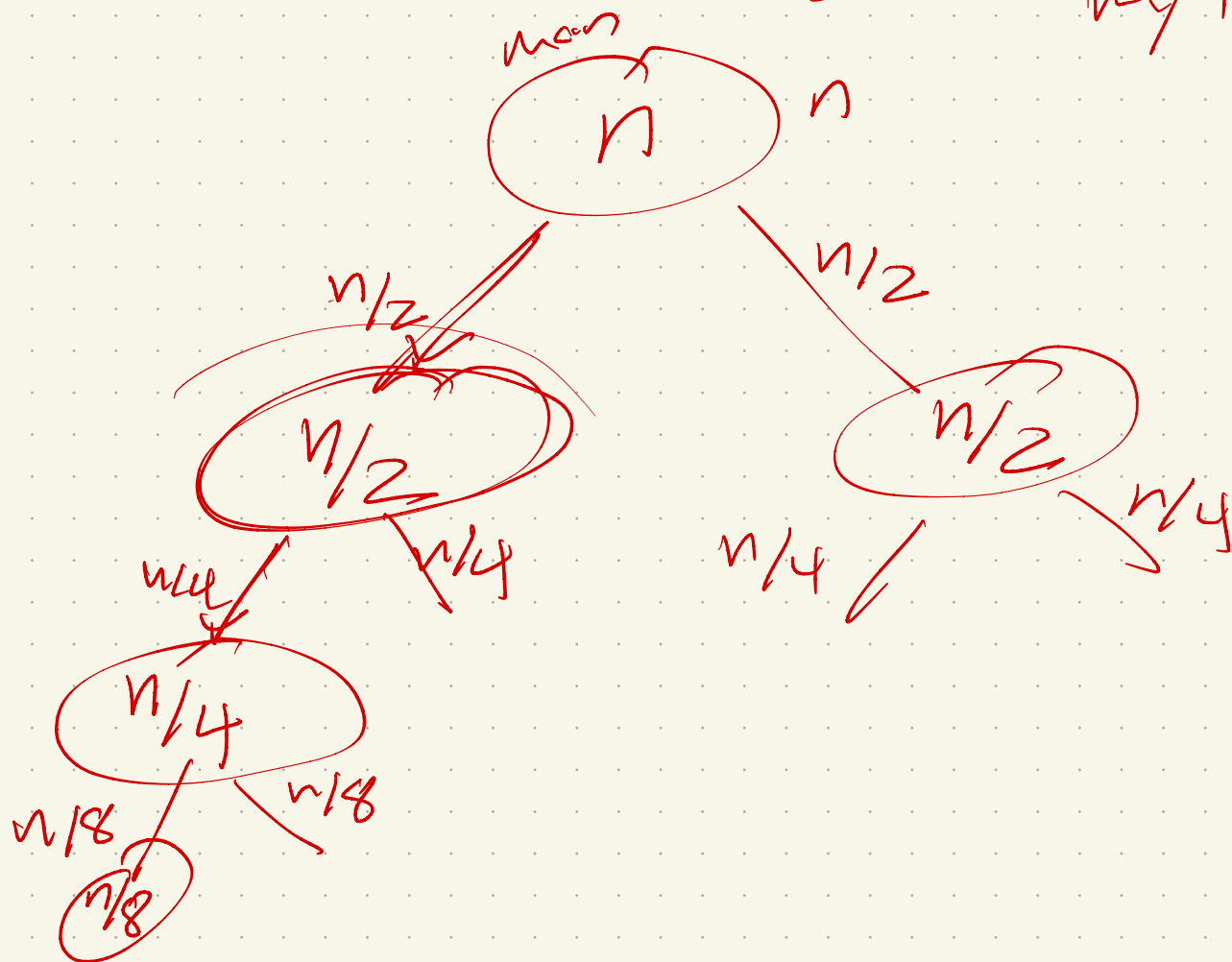
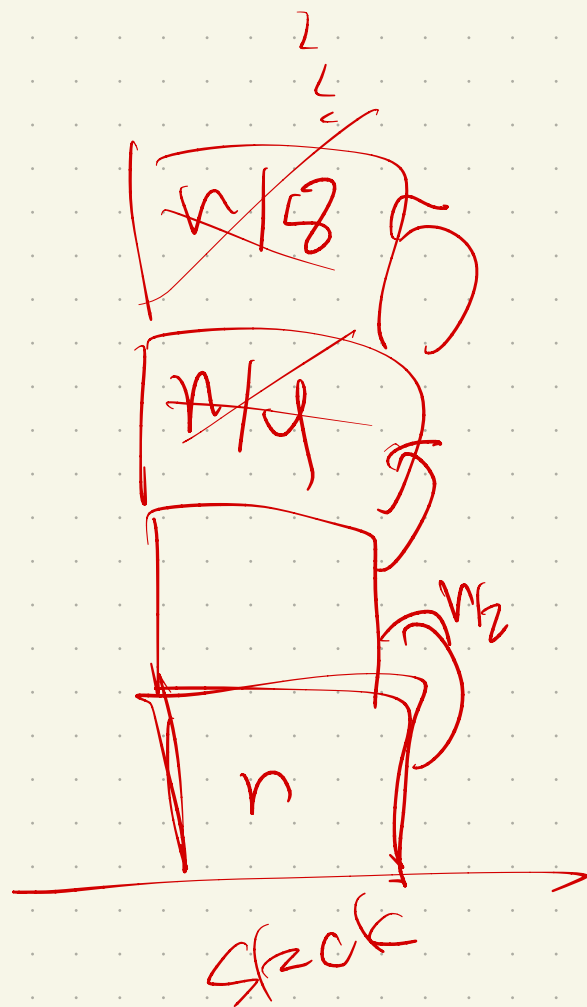
$$\hookrightarrow T(n) = 2T(\frac{n}{2}) + n$$

Recursion "tree":

$$T(n) = 2T(n/2) + n$$

2 rec calls

work
in
my fn



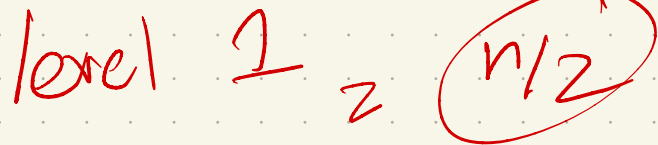
"

1
 $O(1)$

1
 $O(1)$



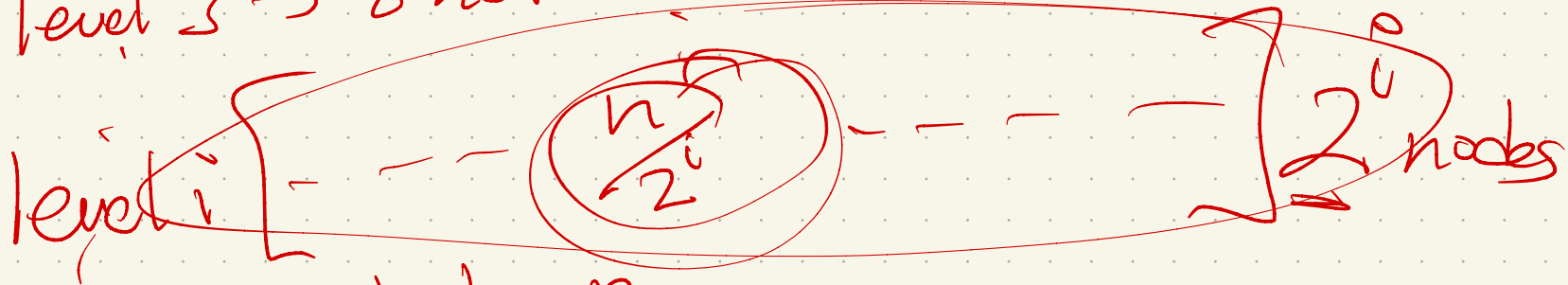
$$\rightarrow T(n) =$$



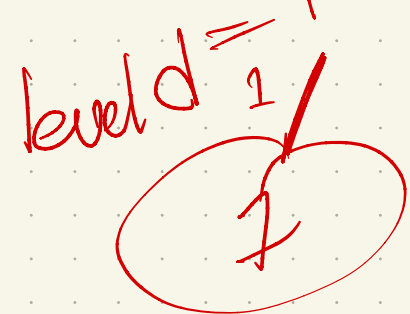
level 2 4



level 3 $\rightarrow$ 8 nodes



level $\log_2 n$



what is d?

$$\frac{n}{2^d} \geq 1$$

$$\Rightarrow n \geq 2^d$$

$$\log_2 n \geq d$$

work
all nodes

$$\sum_{\text{level } i=0}^{\log_2 n}$$

$$T(n) = \sum_{\text{levels}} (\text{work inside a node}) \times (\# \text{ of nodes})$$

$$= \sum_{i=0}^{\log_2 n} \left(\frac{n}{2^i} \right) \cdot 2^i$$

$$= \overset{\text{math!}}{\sum_{i=0}^{\log_2 n}} n = \underbrace{(n + n + \dots + n)}_{\log_2 n}$$

$$= n \log n$$

Idea:
rec

→ tree

"the stack",

Summation of all
work over alg's
lifetime