

Transition: DS + Algs

More logic
Intro to Proofs



Last time: (recap)

- Syllabus Overview
- Intro to Logic

Converse, inverse, & contrapositive:

$$P \rightarrow Q$$

$$Q \rightarrow P$$

$$\Leftrightarrow$$

$$\Leftrightarrow$$

$$\neg Q \rightarrow \neg P$$

$$\neg P \rightarrow \neg Q$$

inverse

n is an integer
 $\in \mathbb{Z}$

Example: IF
 then

n is divisible by 4,
 n is even.

} T

Converse: IF n is even, then n is div by 4.

} F:
 $n=2$
 or
 $n=6$

contrapositive: IF n is odd, then n is not div by 4.

} T

$$n = 2k+1$$

	P	Q	$P \rightarrow Q$	$Q \rightarrow P$
11	T	T	T	T
10	T	F	F	T
01	F	T	T	F
00	F	F	T	T

$\neg Q \rightarrow \neg P$	$\neg Q$	$\neg P$
T	F	F
F	T	F
T	F	T
F	T	T

111

X	Y	$X \rightarrow Y$
T	T	T
T	F	F
F	T	T
F	F	T

Using logical equivalence

We can develop some useful "laws" using logical equivalence!

Example: De Morgan's Law

$$\neg(p \wedge q) = \neg p \vee \neg q$$

P	q	$P \vee q$	$\neg(P \vee q)$	$\neg P$	$\neg q$	$\neg P \vee \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	T
F	T	T	F	T	F	T
F	F	F	T	T	T	T

Why?

not $\Rightarrow$ I am a professor and a computer scientist. $\Rightarrow$

⌊

if either fails, this is F:

I am not a prof, or I am not a comp sci.

Others

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg \neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$x + y = y + x$

$x - y \neq y - x$

$p \rightarrow q \neq q \rightarrow p$

~~while (list is not empty)~~ ^{1=cols} ~~count > 0~~
function call

Predicates $\rightarrow$ ~~TF~~ ^{functions}

propositions which depend on a variable
(or possibly several)

Written $P(x)$: ie $P(x) = "x \text{ is } > 0"$

$\rightarrow Q(x, y) = "x \text{ is a teacher of class } y"$

$R(x) = "x \text{ is prime}"$

Truth value depends on variables:

ie: $P(5) = 5 \text{ is } > 0 \rightarrow T$

$\rightarrow P(-2) = -2 \text{ is } > 0 \rightarrow F$

$Q("Erin", "CSE 50502") = T$

~~$P(x) = 5 + 2x$~~

Quantifiers

$\mathbb{Z}$ $\mathbb{N}$ $\mathbb{R}$

$\mathbb{R}^2 \rightarrow \text{plane}$
 $(x, y) \in \mathbb{R} \times \mathbb{R}$

- Universal quantifier:
 $\forall x P(x)$

$\hookrightarrow$ For all x in domain, $P(x)$ is true

Example:

$\forall x P(x) \rightarrow \forall x \in \mathbb{Z}, x \text{ is prime.}$

Possible values:
should be stated!

T or F

in integers

no divisors
except 1 & x

Every integer is a prime #

Counter example:

To show false, find one element

$x = 4$
 $P(4) = F$ in domains where $P(x)$ is false

• Existential quantifier

$$\exists x P(x)$$

$\exists!$ $\leftarrow$ exist unique

$\hookrightarrow$ "There exists a value x in the domain,
such that $P(x)$ is true"

Example: $\exists x \in \mathbb{Z}$ st x is prime.

How to show it's true? Proof by example.

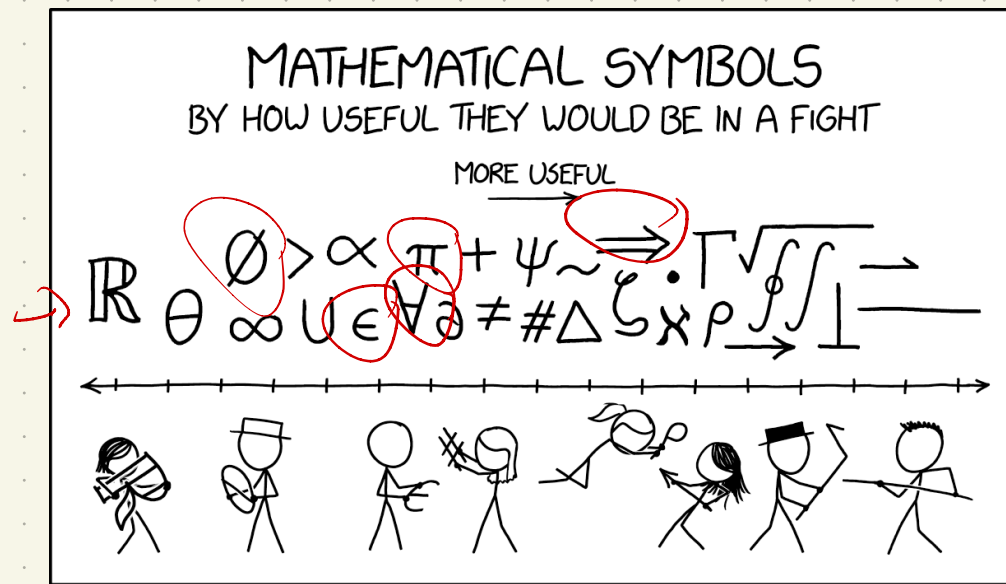
Find the x !

Let $x = 5$, x is prime, $\leftarrow$

(consult math book)

Quick note

At this point, let me pause and acknowledge that math notation can be a lot.



Please stop me with questions!

Using Logical operators

Can combine predicates using all the same logical operators!

$$P(x) = "x + 1 > 2"$$

$$Q(x) = "x < 2"$$

Example:

$$\forall x \in \mathbb{R}, P(x) \vee \neg Q(x)$$

$$\begin{aligned} x &= 2 \\ x &\neq 2 \\ x &\geq 2 \end{aligned}$$

Translate:

For all real numbers x ,
 $x + 1 > 2$ or $x \geq 2$.

Ordering & combining

In short: order matters!

Let $Q(x, y) = "x + y = 0"$

$$\neg (\forall x, \exists y Q(x, y)) \rightarrow \exists x, \forall y \neg Q(x, y) = Q(y, x)$$

Translate *such that*

• $\forall x \exists y$ s.t. $Q(x, y)$:

For all x , there exists y s.t. $x + y = 0$

T: Pick x , set $y = -x$

• $\exists x \forall y$ s.t. $Q(x, y)$:

There is an x s.t. for all y ,
 $x + y = 0$

F

Negations

Consider $P(x) = "x \text{ has taken programming}"$

$\forall \text{ people } x, P(x):$

Everyone has taken programming.

What is $\neg(\forall x P(x))$?

Someone has not taken programming.

$\exists x, \neg P(x)$

Reverse it: what is $\neg(\exists x P(x))$?

$\forall x, \neg P(x)$

They "stack", plus can use laws:

- $\neg(\forall x \forall y P(x, y)) :$
 $\exists x \exists y \neg P(x, y)$

- $\neg(\forall x (P(x) \vee Q(x))) :$

$$\exists x \neg (P(x) \vee Q(x))$$

$$\rightarrow \exists x \neg P(x) \wedge \neg Q(x)$$

Implications & quantifiers

Let $P(x) = "x > 0"$

and $Q(x, y) = "xy \leq 0"$

translate:

$$\forall x \forall y (P(x) \wedge \neg P(y) \rightarrow Q(x, y))$$

Proofs

A theorem (or lemma, proposition, etc.) is a statement that can rigorously be shown to hold true.

Generally in math, something like:

"If n is even, then it is divisible by 2."

" $\forall \varepsilon > 0 \exists \delta > 0$ s.t. if $|f(x) - f(y)| < \varepsilon$,
then $|x - y| < \delta$ "

The sequence of statements giving the argument is a proof.

Consider $p \rightarrow q$:

p	q	$p \rightarrow q$
T	T	
T	F	
F	T	
F	F	

When is it false?

So: to justify $p \rightarrow q$, need to show
not in row _____.

In other words:

if $p = T$, q cannot be F.

First, some base assumptions & defs:

x even:

x odd:

parity: whether even or odd

x divides y , or $x|y$, if:

Other good background:

rational:

composite/prime:

Some other background assumptions:

The rules of algebra.

For example if x , y , and z are real numbers and $x = y$, then $x + z = y + z$.

The set of integers is closed under addition, multiplication, and subtraction.

In other words, sums, products, and differences of integers are also integers.

Every integer is either even or odd.

This fact is proven elsewhere in the material.

If x is an integer, there is no integer between x and $x + 1$.

In particular, there is no integer between 0 and 1.

The relative order of any two real numbers.

For example, $\frac{1}{2} < 1$ or $4.2 \geq 3.7$.

The square of any real number is greater than or equal to 0.

This fact is proven in a later exercise.

Also: addition & mult are commutative!

$$\forall x, y, \quad x + y = y + x$$

Division & subtraction are not!

$$\exists x, y \text{ s.t. } x - y \neq y - x$$

Prove:

Example:

If n is an odd integer,
then n^2 is even.

True or false?

Example:

If n is an even integer,
then n^2 is even.

Proof:

(Assume p is true, and show that q cannot
be false.)

Example: (direct proof)

If x is even and y is odd, then
 $x+y$ is odd

Proof: (assume first part, & directly reason
to get conclusion)

Proof by cases

The book shows some basic proof by cases, but these can get more interesting:

For all $n \in \mathbb{Z}$, $n^2 + n$ is even.

Proof!

Assumptions

Proofs written for publication can assume a lot of background:

Theorem 3.2. Let (X, f) be a generic Reeb graph with vertex set $V(X) = \{v_1, v_2, \dots, v_n\}$. We denote the critical set $S = \{a_1 < a_2 < \dots < a_n\}$ and assume that the vertices are sorted so that $f(v_i) = a_i$. Let $\varepsilon > 0$ be a value such that $2\varepsilon \neq |a_i - a_j|$ for any i, j . Let $W \subseteq V$ be the subset of vertices where each $w \in W$ contributes to a point in Rel_1 which has lifetime at most 2ε . Then there is a bijection

$$\Phi : V(X) \setminus W \rightarrow V(S_\varepsilon(X)),$$

and $S_\varepsilon(X, f)$ is generic.

Even some ktr in class can be intense!

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) = \Theta(f(n))$. ■

Indirect proofs

or "proof by contrapositive"

Recall: $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$

So: if $p \rightarrow q$ seems difficult, we can instead consider showing $\neg q \rightarrow \neg p$ is true

Ex: If $3n+2$ is odd, then n is odd.

Try direct:

Instead:

If $3n+2$ is odd, then

n is odd.

Contrapositive:

Two way implications

Example: For any $n \in \mathbb{Z}$,
 n is odd $\Leftrightarrow n^2$ is odd.

- Here, $\Leftrightarrow$ reads as "if and only if"
- Sometimes written $p \leftrightarrow q$
so $(p \rightarrow q) \wedge (q \rightarrow p)$

How to prove:

Let's try:

For any $n \in \mathbb{Z}$,
 n is odd $\Leftrightarrow n^2$ is odd.

Proof:

$\Rightarrow$:

$\Leftarrow$:

Proof by contradiction: (or indirect proof)

Recall that a contradiction is a logical statement which is always false

Ex: $2=1$
 $x = x+1$

Consider $p \rightarrow q$:

P	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Bad row: want to exclude.

so: $p \wedge \neg q$ is thing to avoid!

Proof by contradiction:

Assume $P \wedge \neg Q$, then show some logical contradiction exists.

Example: If n^2 is even, then n is even.

Another:

Prove that $\sqrt{2}$ is irrational.

Recall: rational means

Proof: