Transition: DS + Algorithms

Linear
recurrences

Worksheet!

```
DigitSum (n):
    if 0 < n < 10:
        return n
    else //(at least 2 digits)
        return [(n mod 10) + DigitSum(n div 10)]
```

one digit smaller than n

c) one digit
↳ return n

b) DigitSum(42)

DigitSum(472)
= DigitSum(47) + 2

on 3052

3	
30	3+0
305	3+5
3052	3+5+2

$$T(n) = O(1) + T(n \text{ div } 10)$$

digits in n → $\log_{10} n$

Recursive algorithms

A function which calls itself
↳ but, with smaller input!

Example: $f_n = f_{n-1} + f_{n-2}$ $\forall n > 1$

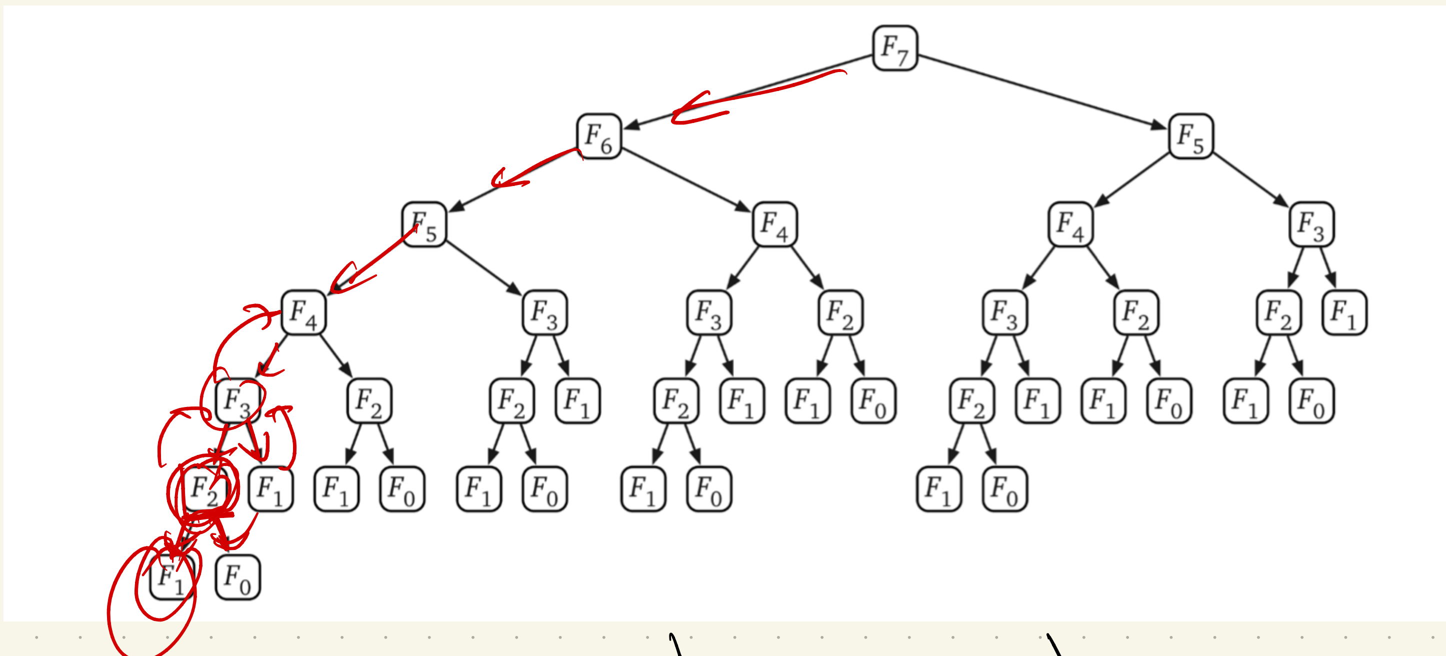
$f_0 = 0$ $f_1 = 1$

0, 1, 1, 2, 3, 5, 8, 13, ...

```
FIB(n):
if n<2:
    return n
else
    return FIB(n-1)+FIB(n-2)
```

Unpacking the recursion

Call Fib(7):



Note:

Not all happening at once!

Check code + think about how programs work...

Why?

Well, for more complex algorithms/structures, can use a recursive definition to build the algorithms

↳ then use induction to prove correctness! (see last reading)

Also → runtime often becomes a recurrence.

So, need to do big-O, but of a recurrence!

Recall: $n! = n[(n-1)(n-2) \cdots 2 \cdot 1]$

$0! = 1$

$1! = 1 \cdot 0!$

How to Compute?

```
FACTORIAL(n):   ← positive integer
    if n = 0
        return 1
    else
        r := FACTORIAL(n-1)
        return n·r
```

$\boxed{1}$, $\boxed{n-2}$, $\boxed{n-1}$, $\boxed{n}$

stack

Correctness:

Proof: induction on n:

Base case: if $n=0$, return 1 ✓

Ind Hyp: assume FACTORIAL$(n-1)$ is correct

Ind Step:

My alg returns $n \cdot \underline{FAC(n-1)}$ ← correct by IH

$\Rightarrow$ returning $n(n-1)! = n!$ ∎

Runtime:

Let $T(k) = $ runtime of function on input k. $= 4 + T(k-1)$ ← order 1

$n \geq 0$

Then:

$$T(n) = 4 + T(n-1)$$

"Unrolling":

$$= 4 + (4 + T(n-2))$$

$$= 4 + 4 + 4 + T(n-3)$$

↳ stops when hit $T(0)$

$\vdots$

$$= \underbrace{(4 + 4 + 4 + \cdots + 4)}_{n} + T(0) = 4n$$

When unrolling fails (& it often does):

```
FIB(n):
if n < 2:
    return n
else
    return FIB(n-1) + FIB(n-2)
```

$F(n)$ = runtime of Fib(n)

Recurrence:

$$F(n) = F(n-1) + F(n-2) + 3$$

$$= \left(F(n-2) + F(n-3) + 3\right) + \left(F(n-3) + F(n-4) + 3\right) + 3$$

$\vdots$

order 2

Dfn A linear homogeneos recurrence has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

where $c_1, \dots, c_d$ are constant

and $c_d \neq 0$

The order of the recurrence is d.

Examples:

$$a_n = 3a_{n-1} + 2a_{n-2} \rightarrow \text{order } 2$$

$$T(n) = 0 \cdot T(n-1) + 0 \cdot T(n-2) + 0 \cdot T(n-3) + 4 \cdot T(n-4)$$

$$T(n) = 4T(n-4) \qquad \text{order } 4$$

$$F_n = F_{n-1} + F_{n-2} \qquad \text{order } 2$$

Examples: Yes or no? Is it linear hom rec?

$P(n) = 1.06\, P(n-1)$ yes $\rightarrow$ order 1

$H(n) = 2H(n-1) + 1$ no $\rightarrow$ added fn

$R(n) = R(n-1) + (R(n-2))^2 \rightarrow$ no, not linear

$a_n = \sqrt{3}\, a_{n-5}$ yes order 5 $c_5 = \sqrt{3}$ c_1, c_2, c_3, c_4 all $= 0$

$f_n = n \cdot f_{n-1}$ n is not constant

$M(n) = 2M(\frac{n}{2}) + n$ no $\leftarrow$ term

$n/2$ is not $n-d$

How to solve:

Generally, these types have solutions which are exponential.

So → look for equations of the form $a_n = r^n$

where r is some constant

If a solution of this form exists, have: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}$

$$\Downarrow$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_d r^{n-d}$$

math!

divide common term: r^{n-d}

$$\Downarrow$$

$$r^d = c_1 r^{d-1} + c_2 r^{d-2} + \dots + c_d r^{0}$$

So: The sequence $a_n = r^n$ is a solution

$$\Updownarrow$$

r is a solution of the characteristic eqn:

$$r^d - c_1 r^{d-1} - \dots - c_d = 0$$

How to find r? math!

Ex: $P(n) = 1.06 \cdot P(n-1)$ order 1

$\hookrightarrow r^1 = 1.06 \Rightarrow r - 1.06 = 0$

$r = 1.06$ $\quad P(n) = c \cdot (1.06)^n$

Ex $a_n = a_{n-1} + 2a_{n-2}$ order 2

$r^2 - 1r - 2 = 0$ char eqn

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$ (or use computer)

roots 2 & -1

$a_n = c_1 \cdot 2^n + c_2(-1)^n$

Ex: $F_n = F_{n-1} + F_{n-2}$ order 2

$$r^2 - r - 1 = 0$$ char eq^n

$$\cancel{(r \quad 1)(r \quad 1)}$$

$a=1$ $c=-1$
$b=-1$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$

golden ratio

$$= \frac{1 \pm \sqrt{5}}{2} = \phi$$

$$\frac{1+\sqrt{5}}{2}$$

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ex: $B(n) = 2B(n-1) + B(n-2)$

... this is a bit different!

Finding general solutions

- If r is a non-repeated root of the char. eqn, then r^n is a solution to the recurrence.
- If r is a repeated root with multiplicity k then $r, n \cdot r^n, \ldots n^{k-1} r^n$ are <u>all</u> solutions
- Any linear combination will satisfy the recurrence, but need to use initial values to find constants

Huh?

Examples

- $P(n) = 1.06\,P(n-1)$

  $P(0) = 10{,}000$

  $x - 1.06 = 0 \Rightarrow$ root is $x = 1.06$

  so $P(n) = c\cdot(1.06)^n$

  what is c?

- $A(n) = A(n-1) + 2A(n-2)$

& $A(0) =$

$A(1) =$

$\Rightarrow x^2 - x - 2 = 0$ had roots

$x = 2$ and $x = \underline{-1}$

So: $A(n) = C_1 \cdot 2^n + C_2 \cdot (-1)^n$

- Suppose a characteristic equation for $C(n)$ simplifies to:

$$(x-2)^3(x-5)^2(x+9)=0$$

Then roots are:

and $C(n) =$

So:

We can find runtimes, as long as the recurrence is linear & homogeneous!

Q: What if it's not??

Two others we'll study here, since they are useful in computing:

- linear inhomogeneous
- divide & conquer

Linear inhomogereous recurrences

These are just linear ones, with an added function $g(n)$ at end:

$$f(n) = c_1 f(n-1) + \cdots + c_d f(n-d) + g(n)$$

Examples:

$$F(n) = F(n-1) + F(n-2) + 1$$

$$A(n) = 4A(n-1) - 2^n + n^2$$

Next time: How to solve?