Transition: DS + Algorithms

Induction
+ Recursion

$p \wedge q = q \wedge p$, $p \vee q = q \vee p$

Last time:

$$A-(B\cup C) \overset{\subseteq}{=} (A-B)\cap(A-C)$$

$\subseteq$: Take $x \in A-(B\cup C)$

by dfn of $-$, $\Rightarrow$ $x\in A$ and $x\notin B\cup C$

$\neg[x\in B \cup C]$ or

DeMorgans

$\Rightarrow$ $x\in A$ and $[x\notin B$ and $x\notin C)$

or

All ands! Rearrange:

$x\in A$ and $x\notin B \Rightarrow x\in A-B$

$x\in A$ and $x\notin C \Rightarrow x\in A-C$

So: $x\in A-C$ and $x\in A-B$

$\Rightarrow x\in (A-C)\cap(A-B)$

Back to induction:

Worsheet?

Induction is not just for numbers! $P(n-1) \to P(n)$

(need some basic axioms $n \to n+1$)

Prove:

If S is a finite set with $\underline{k}$ elements, then $\underbrace{P(S)}$ has size 2^k.

set of all subsets of S

$S = \{1, 2, 3\}$

$P(S) = \{\emptyset, \{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

$|P(S)| = 8 \quad |S| = 3$

$2^3 = 8$ ✓

$S' = \{1,2\}$

$P(S') = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$+ \{3\}$

Cont Induction on k = size of set S

BC ϕ $|\phi| = 0$ $\phi \subseteq \phi$

$P(\phi) = \{\phi\}$

$|P(\phi)| = 1$ $2^0 = 1$ ✓

size 1: $|S| = 1$: $\{\phi, S\} = P(S)$ ✓

$2^1 = 2$

$\boxed{\text{IH: For } |T| = k \geq 0 \text{ have } |P(T)| = 2^k}$ sets in it.

IS: Consider $|S| = k+1$

Take any $x \in S$ (Must be something since past base case)

Let $T = S - \{x\}$

$\Rightarrow |T| = k$ so by IH,

$$|P(T)| = 2^k.$$

Convince you about $P(S)$:

① $T \subseteq S$, so every element in $P(T)$ is also in $P(S)$.

$$|P(S)| \geq 2^k$$

② Anything in $P(T)$ can't have x

$\hookrightarrow \{x\} \cup V$ for $V \in P(T)$ is also in $P(S)$

$$|\{x\} \cup V \text{ for } V \in P(T)| = |P(T)|$$

Any set $V \in P(S)$ either has x or doesn't

$\underbrace{\{x\} \cup V}_{|2^k|}$, $V \in P(T)$, $T = S - \{x\}$ "rule of sum"

from $P(T)$ $|2^k|$

$$|P(S)| = 2^k + 2^k$$

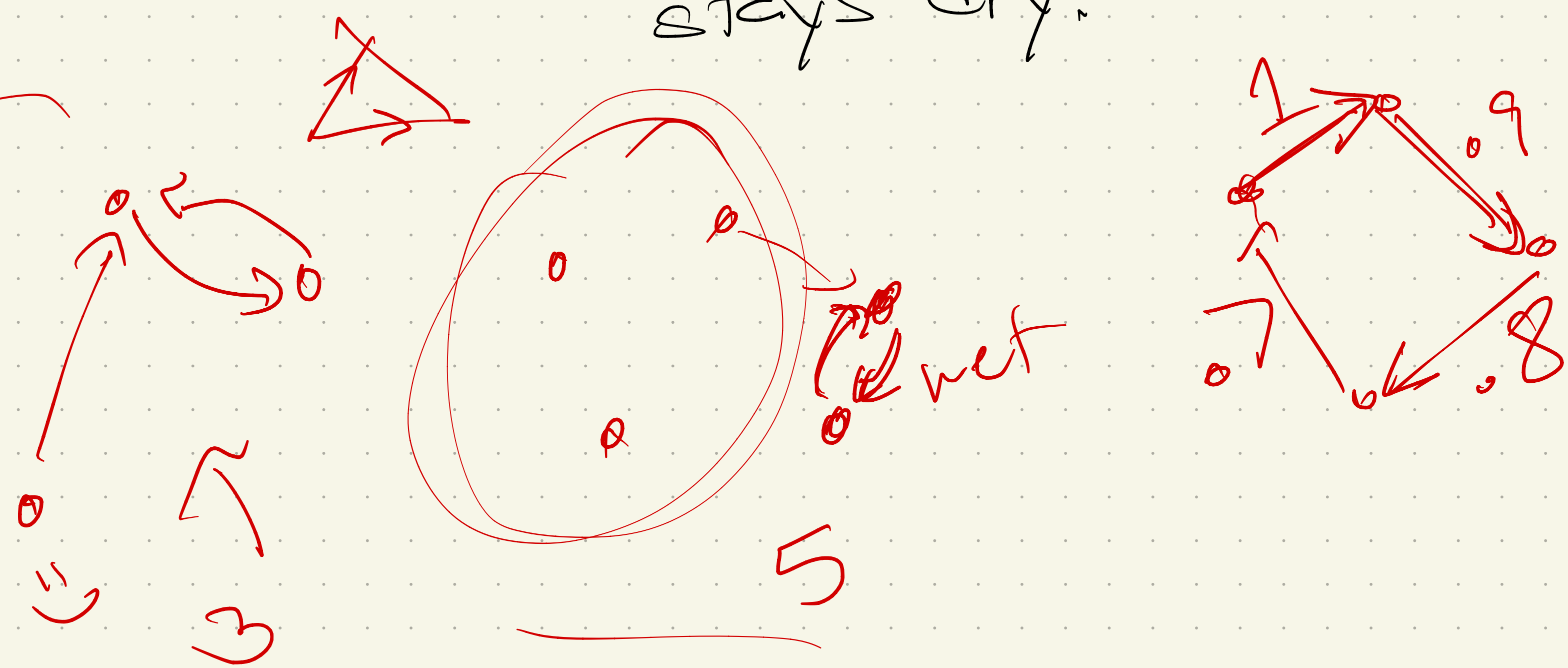
$$= 2^1 \cdot 2^k = 2^{k+1}$$

$\boxtimes$

Another

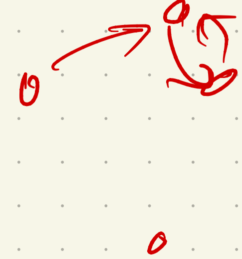
Suppose n friends have a water balloon fight. Each moves to some location (all distinct), and then throws the balloon at their nearest person. (assume unique closest)

Claim:
If n is odd, then at least 1 person stays dry.



Cont:

BC: ~~$n=3$~~ or $n=1$



some pair are closest (+ throw at each other)

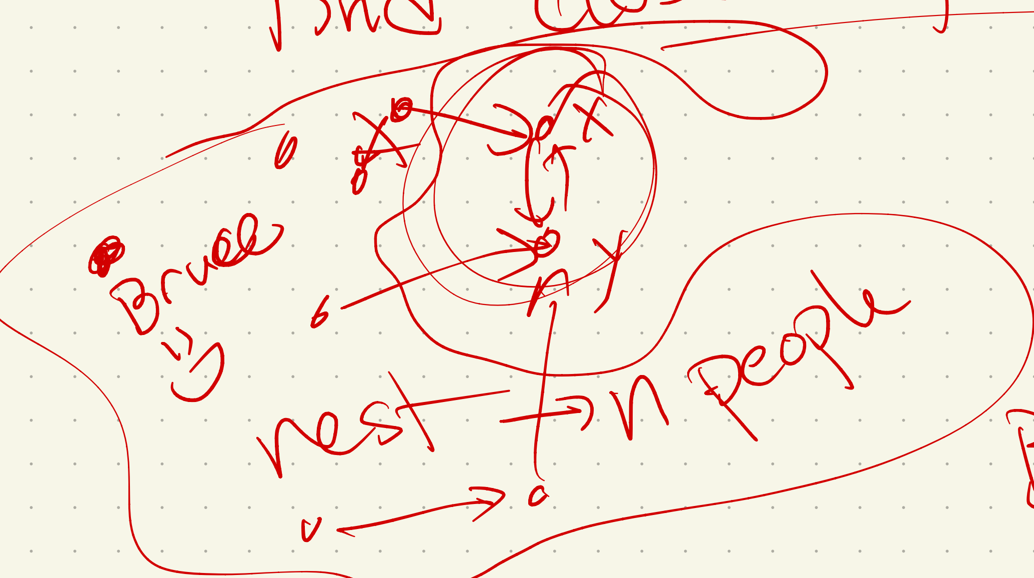
person 3 is dry

IH: True for n odd: n people water fight, 1 always is dry.

IS: Consider $n+2$ odd:

$n+2$ people:

Find closest pair x + y and remove them



→ smaller fight has n, so some is dry by IH

Consider adding x + y:

Bruce is still dry.

Why is Bruce dry?

Before (w/out x + y) no one thought he was closest.

After adding x + y:

Bruce's dist to everyone is same, except x + y

x + y are global closest pair

↳ they won't throw at Bruce.

☑

Gossip Problem:

- n total people, each know a secret
- Every time 2 people call each other, they share every secret they know

How many phone calls are needed for everyone to know <u>all</u> n secrets?

Try some small ones:

Claim: If $n \geq 4$, then $2n-4$ suffice.

<u>proof</u>: induction on $n = \#$ of people

<u>base case</u>: 4 people.

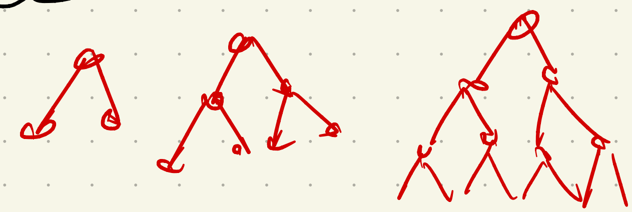
<u>Ind hyp</u>: for $n-1$ people, can solve with $2(n-1)-4$ calls

<u>Ind step</u>:

Next time: Recursion → recursive defns

Example: $F_n = F_{n-1} + F_{n-2}$, $n \ge 2$ or $F_{n+2} = F_{n+1} + F_n$, $n \ge 0$

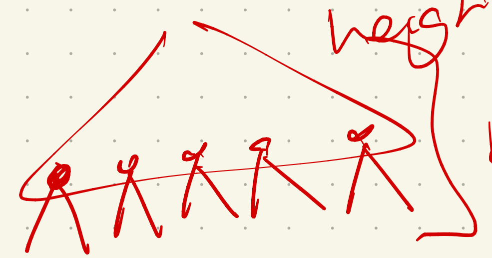
Perfect binary trees can be defined recursively:

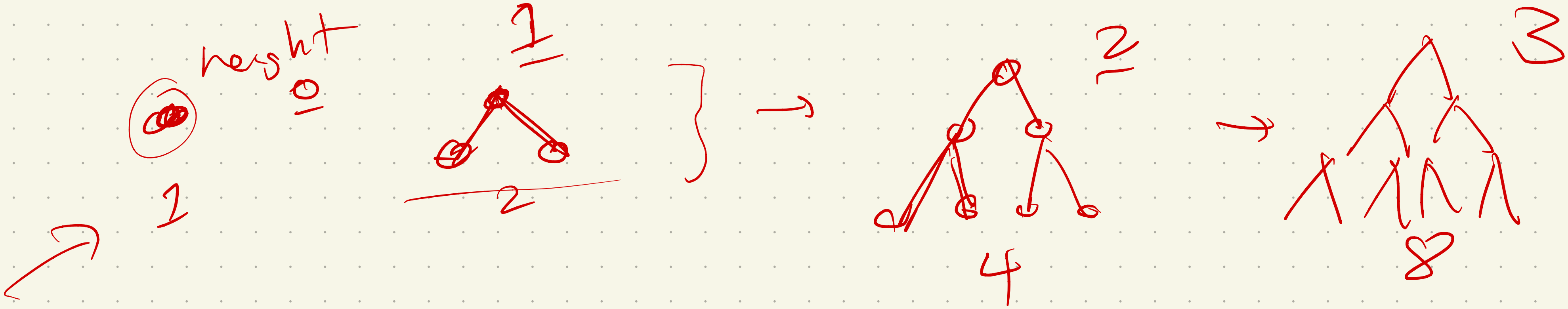


Basis: Single vertex, height $= 0$

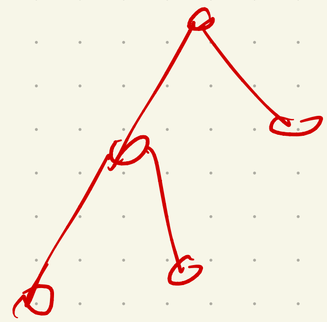
Recursive rule: given any perfect binary tree T of height $k \ge 0$, build one of height $k+1$:

Take every leaf in T, add 2 children to that leaf.





Not:



not perfect

Base case: 1 leaf

Fact/obs: each time, # of leaves doubles.

leaves: $\Rightarrow 2^h$ leaves

Theorem:

Every perfect binary tree has size $= 2^{\cancel{k}\,h+1}-1$ vertices ~~for some positive $k \in \mathbb{Z}$.~~ where h is height

Induction on height.

Base case: height 0

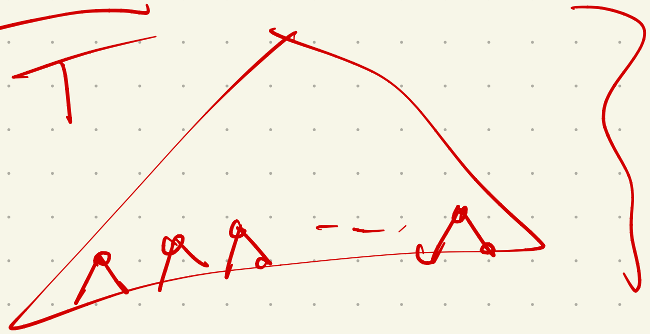
1 vertex. $1 = 2^{1} - 1$ ✓

$k = 1 = h+1$

IH: Assume true for height h

IS: Consider tree T of height $h+1$.

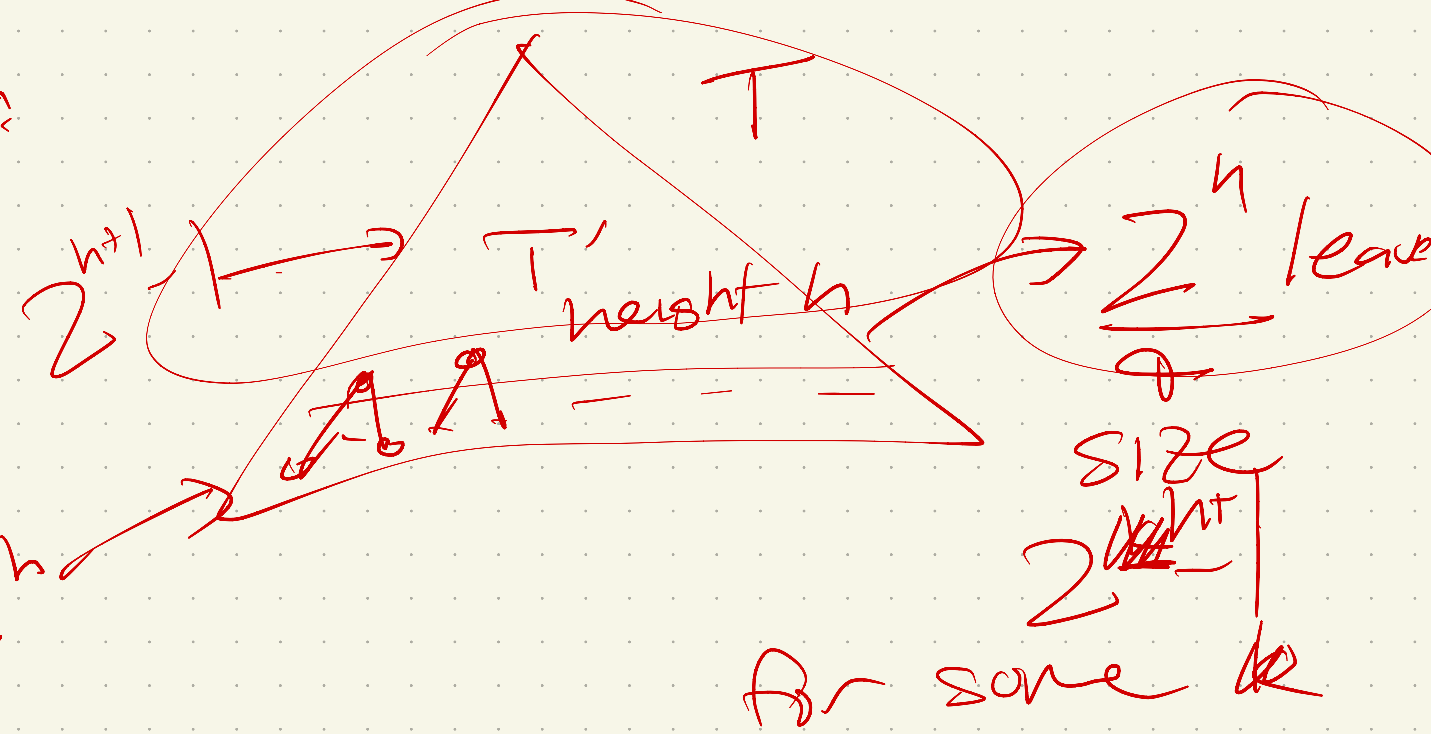
T



$h+1$ Built from some T' of height h

how many leaves did it have?

use fact:



$$|T'| = 2^k - 1$$

$$|T| = |T'| + 2\cdot(\text{\# leaves in } T')$$

$$= (2^{h+1}-1) + 2\cdot 2^h$$

$$= 2^{h+1} - 1 + 2^{h+1} = 2\cdot 2^{h+1} - 1 = 2^{h+2} - 1$$

Every perfect bin tree of height h has

$$\underbrace{2^0 + 2^1 + \cdots + 2^h}$$ vertices

$$\sum_{i=0}^{h} 2^i = 2^{h+1} - 1$$

1 0 0 0 0 0

$$\underbrace{\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{array}} = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$