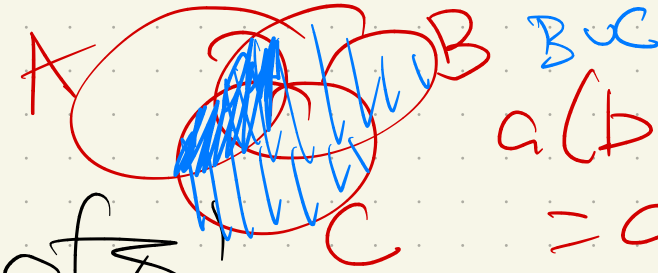
Transition: DS → algorithms

Sets recap
Induction

First:

Set proofs!



$a(b+c)$
$= ab + ac$
in math
↳ distributive

#2: A B + C,

$$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$$

Assume $x\in A\cap(B\cup C)$

by defn of $\cap$ $\Rightarrow$ $x\in A$ and $x\in B\cup C$

$\Rightarrow$ $x\in A$ and $[x\in B$ OR $x\in C]$

(stuck) by dist. law on logic,

$\Rightarrow$ $[x\in A$ and $x\in B]$ OR $[x\in A$ and $x\in C]$

$\Rightarrow$ $x\in A\cap B$ or $x\in A\cap C$ $\Rightarrow$ $x\in$ $(A\cap B)\cup(A\cap C)$

2: $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in A \cap B$ OR $x \in A \cap C$

$\Rightarrow (\overbrace{x \in A \text{ and } x \in B}^{p})$ OR $(\overbrace{x \in A \text{ and } x \in C}^{q})$

#1 apply logic distr. law!

$\Rightarrow x \in A$ and $(x \in B$ OR $x \in C)$

by dfn $\Rightarrow x \in A \cap (B \cup C)$

#2 $\Rightarrow$ well, need 1st or 2nd clause to hold for the OR to be true $p \vee q$

Note $x \in A$ is in both! so, if $x \notin A$, it's not true already.

$\Rightarrow x \in A$ then for p or q to be true, need at least one of $x \in B$ or $x \in C$.

#3) $A - (B \cup C) = (A-B) \cancel{\cap} \cup (A-C)$

oops!

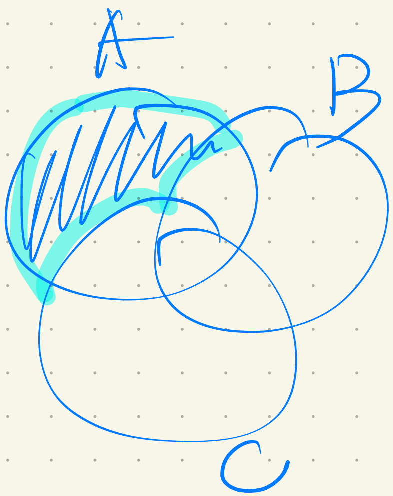
Assume $x \in A-(B \cup C)$

by dfn $\Rightarrow$ $x \in A$ and $x \notin (B \cup C)$

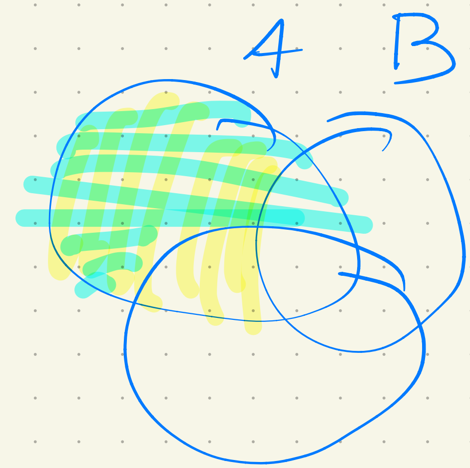
$\Rightarrow x \in A$ and $[x \notin B$ or $x \notin C]$

try #2:

$(x \in A \qquad) \cap (x \in A \qquad)$



stay tuned!



Induction

A very powerful proof method, used for statements of the form:

base case

$$\forall n \geq c,\ P(n)$$

$\forall k \geq 1,$ if $P(k)$ then $P(k+1)$

Strategy:

base case

① Show $\boxed{P(1)}$ is true

$k-1=n$ $\quad$ $k=n+1$

② Show $\boxed{\forall k>1,\ P(k-1) \to P(k)}$

if $P(k-1)$ is true, then $P(k)$ is true

$P(44)$

Why?

$P(1)$ is true

by ② $\Rightarrow P(2)$ is true

$\Rightarrow P(3)$ is true $\Rightarrow P(4) \Rightarrow \ldots$

$P(2)$

$\Rightarrow P(3)$

$\Rightarrow P(4)$

$\ldots$

Alternatives:

Show true for all numbers directly.

Great if it works! $\Rightarrow 1+2+3+\cdots+n$

But $\rightarrow$ why is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$?

Sum of all $i=1$ to n

Sometimes, hard to know how to tackle

$n \rightarrow n-2 \rightarrow n-4 \rightarrow n-6$

$\hookrightarrow$ this is just another method!

Pair!

$$\underbrace{1 + 2 + 3 + \cdots + (n-2) + (n-1) + n}_{=n+1} = (n+1)\frac{n}{2}$$

$n+1$

$n+1$

dreft:

$$\cancel{1} + 2 + 3 + 4 + \cdots + (n-3) + (n-2) + (n-1) + \cancel{n}$$

remove $1+n \rightarrow$ left $\underbrace{2+3+4+\cdots+(n-1)}_{n-2 \text{ terms}}$

$\hookrightarrow$ remove $2+(n-1) = n+1$

$3+4+\cdots+(n-2) \rightarrow$ length $n-4$ terms

$\hookrightarrow n-6 \rightarrow n-8 \dashrightarrow$ when 0 terms $\rightarrow n/2$

$$\underbrace{10 \to 8 \to 6 \to 4 \to 2 \to 0}_{10/2}$$

$$1 + \cdots + n = (n+1)\left(\frac{n}{2}\right) = \frac{n(n+1)}{2}$$

each pair

pairs

Writing inductive proofs

3 pieces: $\forall n, P(n)$:

- Base case:
  show ~~true~~ true for some base case: $P(c)$ 1
- Inductive Hypothesis:
  - assume $P(k)$ is true
- Inductive step:
  - use IH to show $P(k+1)$ is also true

Recap: $\boxed{\forall n,}\ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Side bar: why should we care?

forloops: for $i \leftarrow 1$ to n

for $j \leftarrow 1$ to i

~~average~~

Proof: by induction on n

Base case: try $n=1$ $P(1)$

$\sum_{i=1}^{1} i = 1$ NTS $= \frac{1(1+1)}{2} = 1$

✓

$\forall k, \boxed{P(k)} \to P(k+1)$

Ind Hyp:

Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Ind Step:

Show $\sum_{i=1}^{k+1} i = \boxed{\frac{(k+1)(k+2)}{2}}$ ✓

direct: start with $\sum_{i=1}^{k+1} i$

$$\sum_{i=1}^{k+1} i = \underbrace{1+2+\cdots+k}+(k+1)$$

$$= \sum_{i=1}^{k} i + (k+1) \quad \text{apply I.H.}$$

$$= \frac{k(k+1)}{2} + (k+1) = (k+1)\left[\frac{k}{2}+1\right] = \frac{(k+1)}{2}(k+2) \quad \frac{k+2}{2}$$

∎

Geometric series:

If $r \neq 0$, $\displaystyle\sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot r^{n+1} - a}{r-1}$

<u>Proof</u>: Fix $r \neq 0$, & induct on n:

Claim: $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Proof: induction on n

Base case: $n = 1 \longrightarrow P(1)$ T

show div by 3

$1^3 - 1 = 0$

by dfn $\Rightarrow$ need $k \in \mathbb{Z}$ s.t. $3k = 0$

Let $k = 0$, then $3 \cdot 0 = 0$ ✓

IH: Assume $\overset{P(n)}{n^3 - n}$ is div by 3

IS: Consider $\overset{P(n+1)}{(n+1)^3 - (n+1)}$

IH: Assume $\overbrace{n^3-n}$ is div by 3

$\iff \exists k \in \mathbb{Z}$ s.t. $n^3-n=3k$

IS: Consider $(n+1)^3-(n+1) \overset{?}{=}$ div by 3!

$(n+1)(n+1)$

$(n^2+2n+1)(n+1)$

$$= (n^3+3n^2+3n+1) - (n+1)$$

$$= n^3+3n^2+2n+\overbrace{(n-n)}^{0}$$

$$= (n^3-n)+n+3n^2+2n$$

$$= \underbrace{(n^3-n)}_{\text{div by } 3} + \underbrace{3n^2+3n}$$

$$= 3k+3(n^2+n) = 3\underbrace{(k+n^2+n)}_{\in \mathbb{Z}}$$

$$= n^3 + 3n^2 + \underline{2n} \qquad + \underline{(n-n)}$$

$$= n^3 + 3n^2 + (\underline{3n - n})$$

$$= \underbrace{(n^3 - n)}_{\text{div by 3 by IH}} + 3n^2 + 3n$$

$$= 3(n^2 + n) \quad \in \mathbb{Z}$$

$$\Rightarrow \text{div by } 3$$

$\Rightarrow$ whole thing is:

$(n+1)^3 - (n+1)$ is div by 3 ∎

$$(n+1)^3-(n+1)$$

$$=(n+1)\left((n+1)^2-1\right)$$

$$=(n+1)(n^2+2n)$$

$$=(n+1)(n)(n+2)=3k$$

$(n+x)$

$(n+x+1)$

$x \bmod 3 = 0$ remainders

1

2

if $x \equiv 0 \mod 3$

$x+1 \equiv 1 \mod 3$

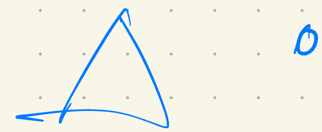
$x+2 \equiv 2 \mod 3$

Next time:

Induction <u>not</u> on numbers

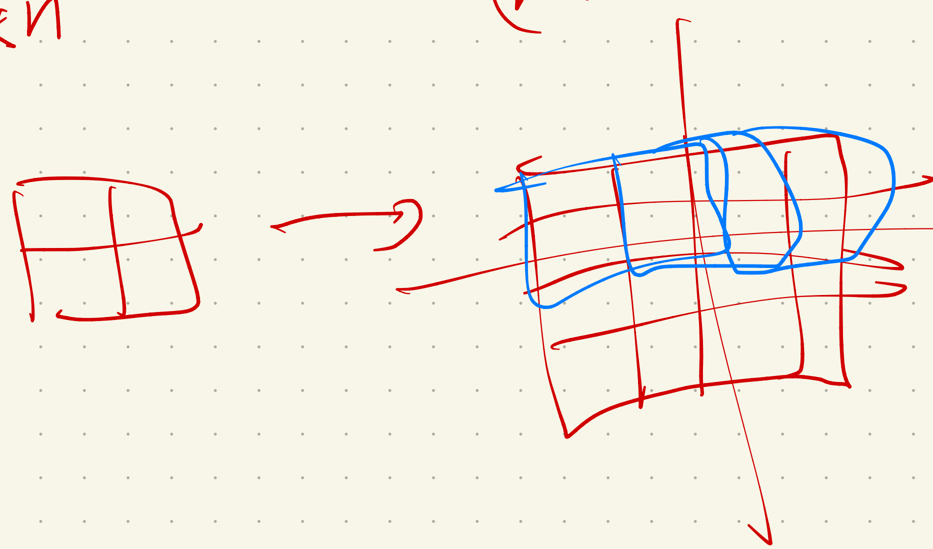
(I delayed the next 2 readings so we can do more induction.)

$n \to n+1$



$(n+1)*(n+1)$

$n*n$



3 vertex graph

4 vertex graph