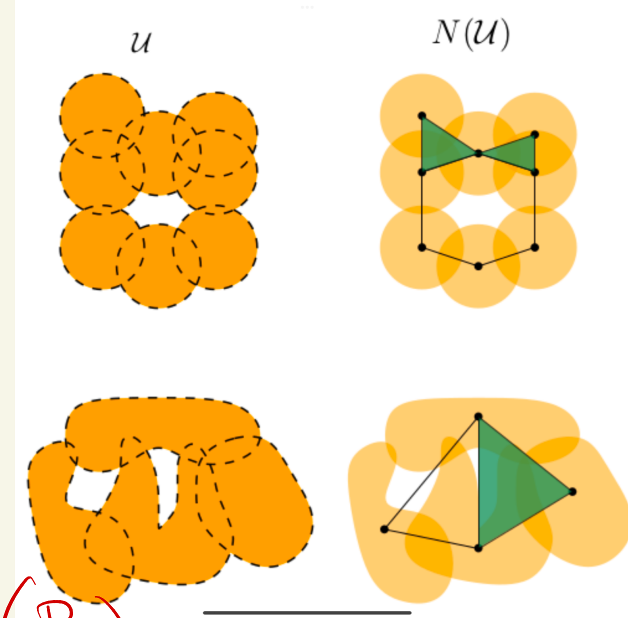
TDA - fall 2025

Voronoi diagrams
α-shapes
Chain complexes

Last time:

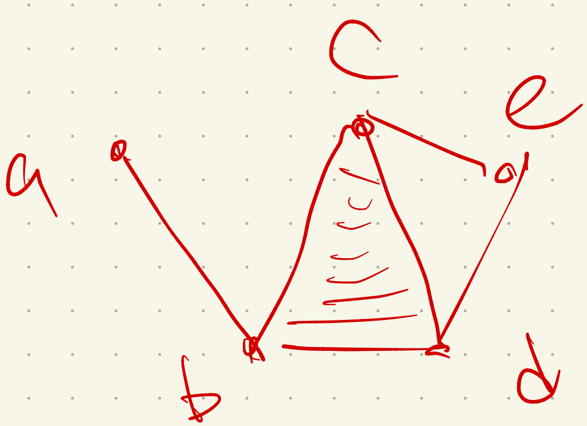
Nerves make good approximations of a space if $\cap$'s are contractible

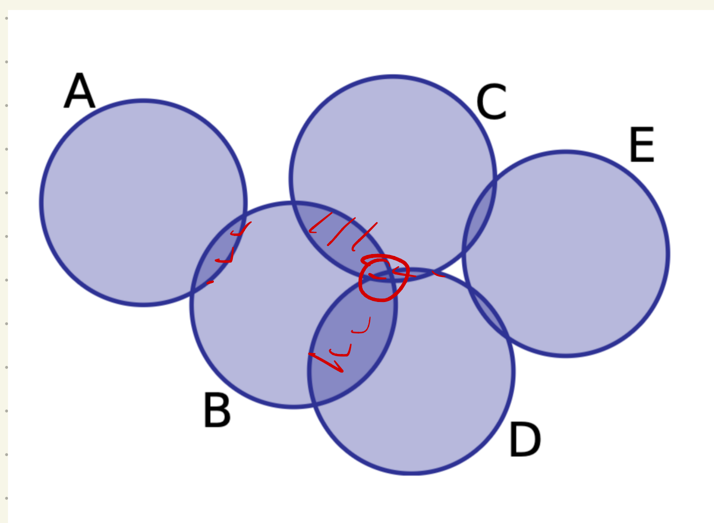


$$C_\varepsilon(B) \subseteq R_\varepsilon(B) \subseteq C_{2\varepsilon}(B)$$

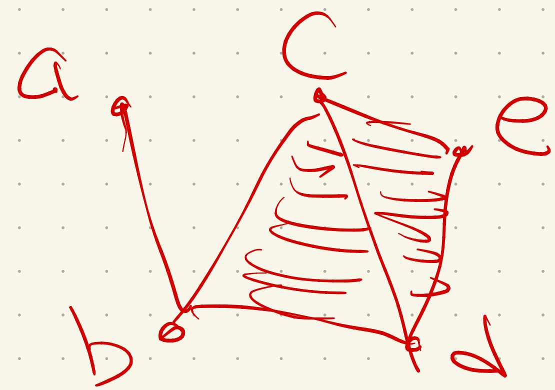
We saw 2: Čech & Rips complexes

Čech:





Rips:



Voronoi diagrams

Given a set of points P in $\mathbb{R}^d$, the Voronoi cell for site $p \in P$ is

$$V_p = \{x \in \mathbb{R}^d \mid d(x,p) \leq d(x,q) \ \forall q \in P)$$

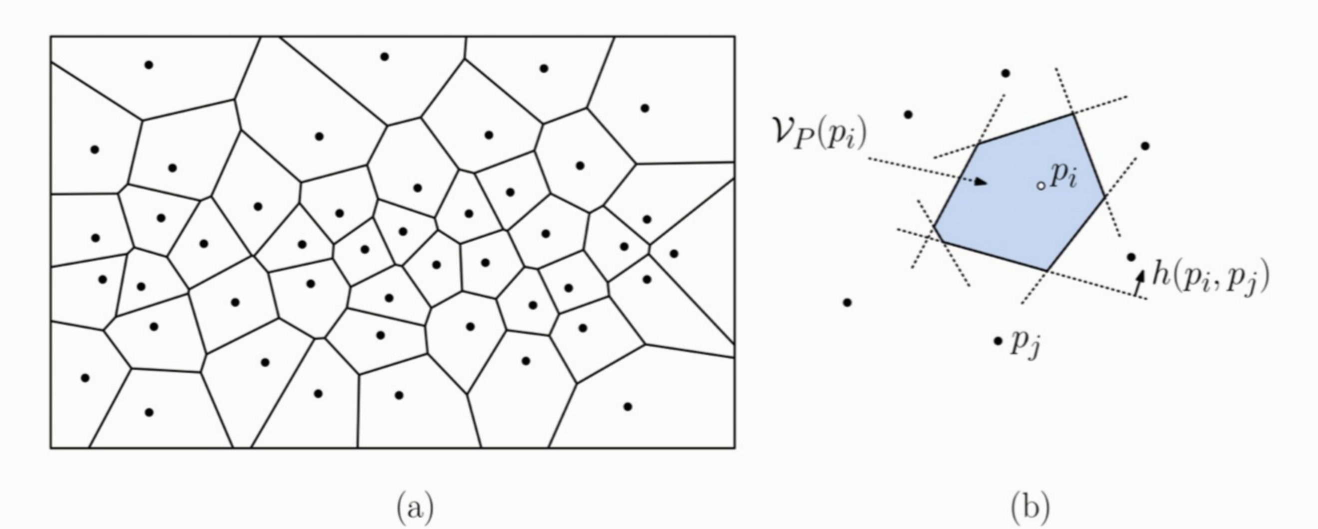


Fig. 55: Voronoi diagram Vor(P) of a set of sites.

This tesselates $\mathbb{R}^d$, & the collection of cells is the Voronoi diagram $\mathrm{Vor}(P) = \{V_u \mid u \in P\}$

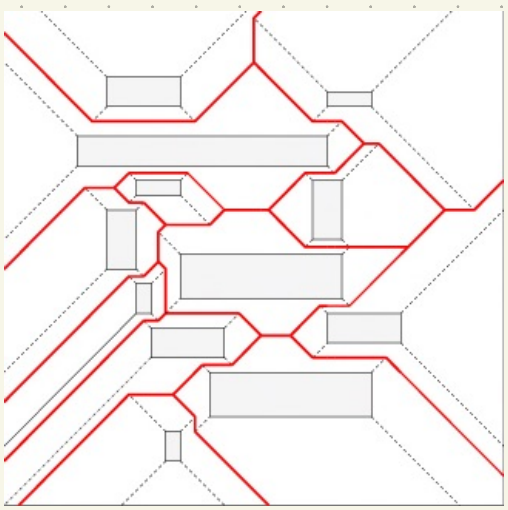
Why?

Super useful!

- Closest point queries
- Shape analysis
- Clustering

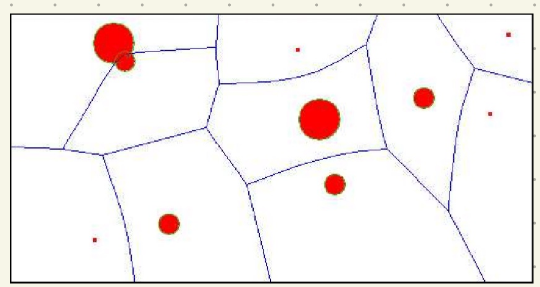
$\Rightarrow$ **lots** of code

Even variants for other metrics on $\mathbb{R}^d$!



ℓ_1 distance, polygons

weighted Voronoi



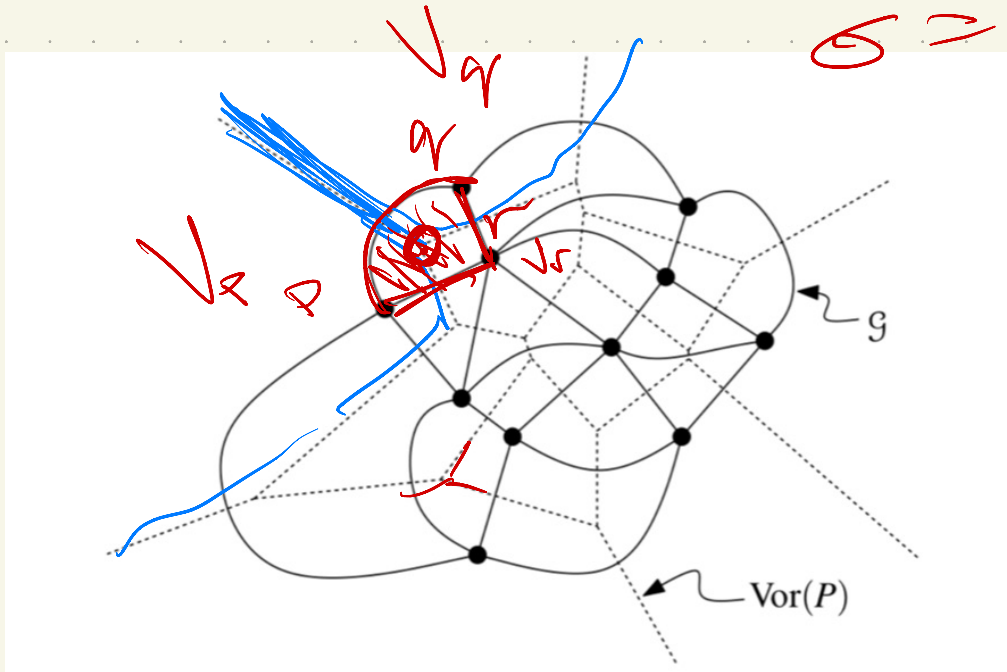
Why we care

The Delaunay complex of $P \subseteq \mathbb{R}^d$ is the nerve of the Voronoi diagram!

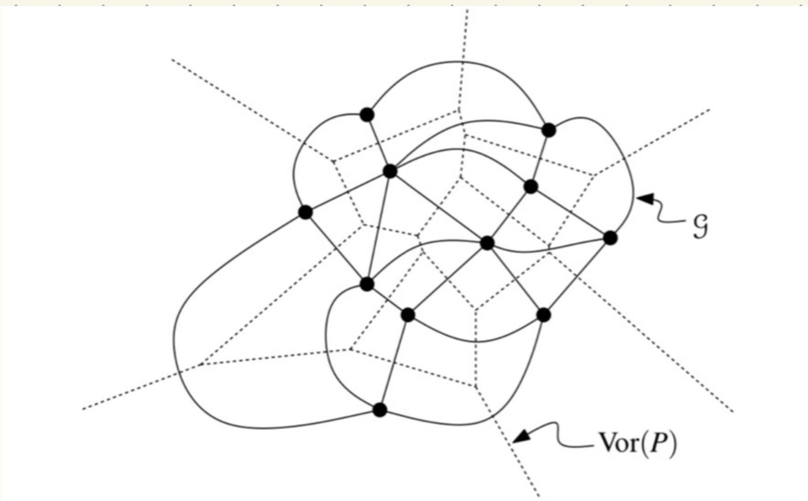
$$\mathrm{Del}(P) = \left\{ \sigma \subseteq P \;\middle|\; \bigcap_{u \in \sigma} V_u \neq \emptyset \right\}$$

$\sigma = \{p, q, r\}$

Note: still an abstract simplicial complex!

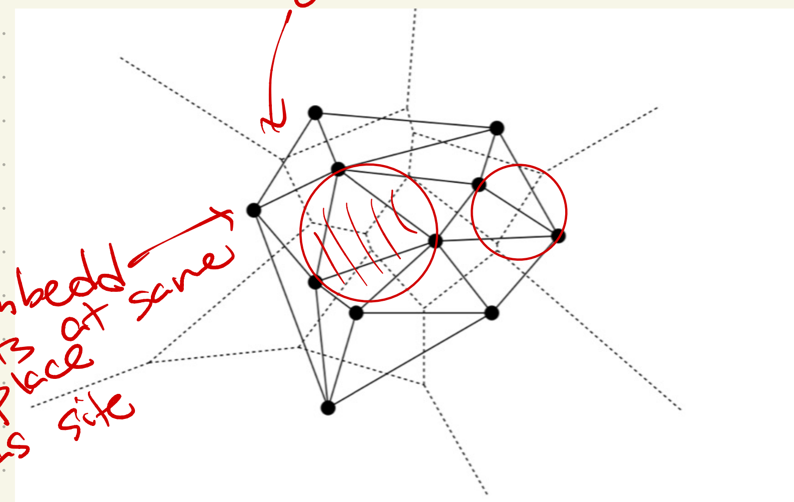


Fact: The "obvious" embedding of Del(P) gives a geometric simplicial complex:



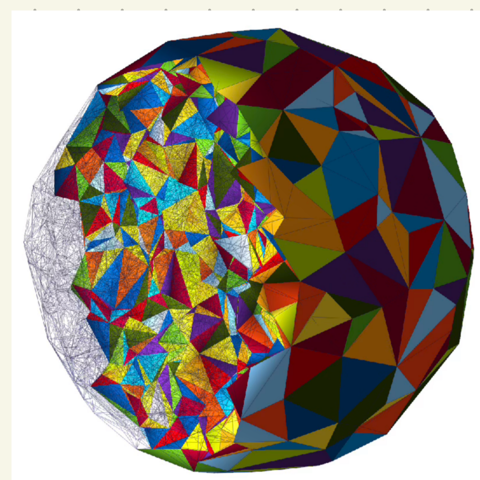
→

draw edges orthogonally

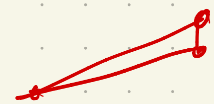


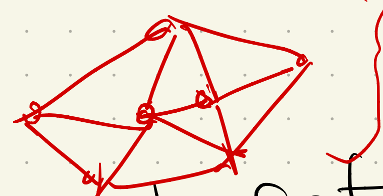
embed points at same place as site

Note: no parameter r here – Del(P) & Vor(P) are fixed.



Why is it nice?





A triangulation of a point set $P \subset \mathbb{R}^d$ is a geometric simplicial complex with point set P whose simplices tesselate the convex hull of P.

Among all triangulations, Del(P):

1) minimizes the largest circumcircle for $\triangle$'s in the complex (in $\mathbb{R}^2$)

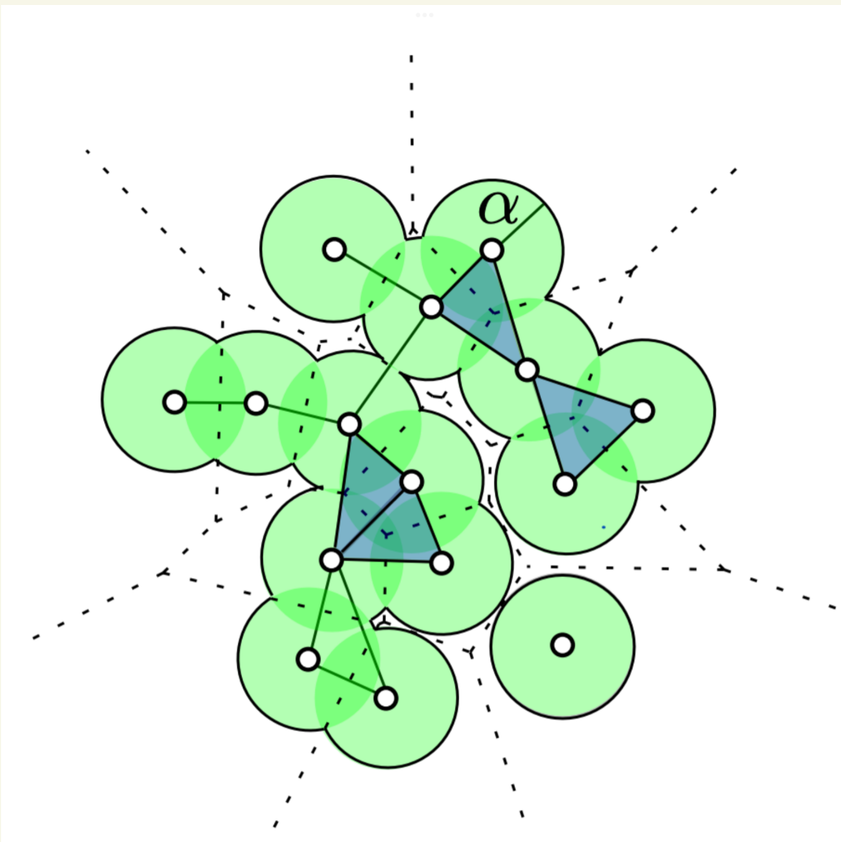
2) maximizes the minimum angle of $\triangle$'s in the complex (in $\mathbb{R}^2$)

3) All minimum enclosing balls of $\mathbb{R}^d$ simplices are empty, & largest is minimized

Adding r back in:

Let $D_p^{\alpha} := \{x \in B(p,\alpha) \mid d(x,p) \le d(x,q) \ \forall q \in P\}$

$= B(p,r) \cap V_p$



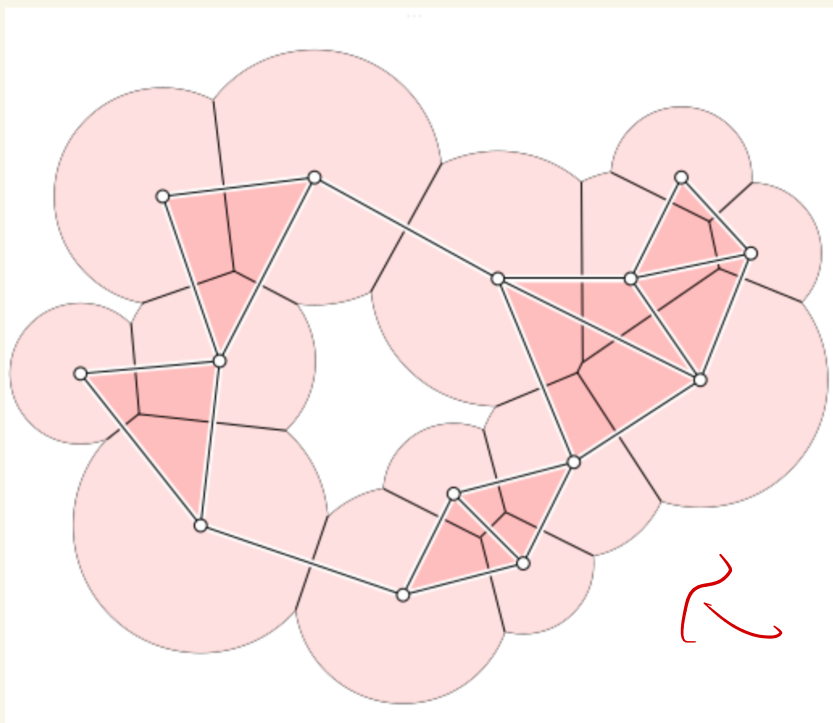
The alpha complex

$$\mathrm{Del}^{\alpha}(P) = N(\{D_p^{\alpha} \mid p \in P\})$$

Properties

- $Del^\alpha(P) \subseteq Del(P)$
- $Del^\alpha(P) \subseteq \check{C}(r)$
- $Del^\alpha(P)$ has the same homotopy type as the union of balls of radius r.

actually def retract



The book covers 2 other types of complexes: witness complex & graph induced complex.

Both describe ways to "sparsify" data:

Find a "good enough" subsampling of a point set P:

take $Q \subseteq P$ & define a simplicial complex <u>on Q</u> ← small

(but using P to build simplices)

Witness Complex

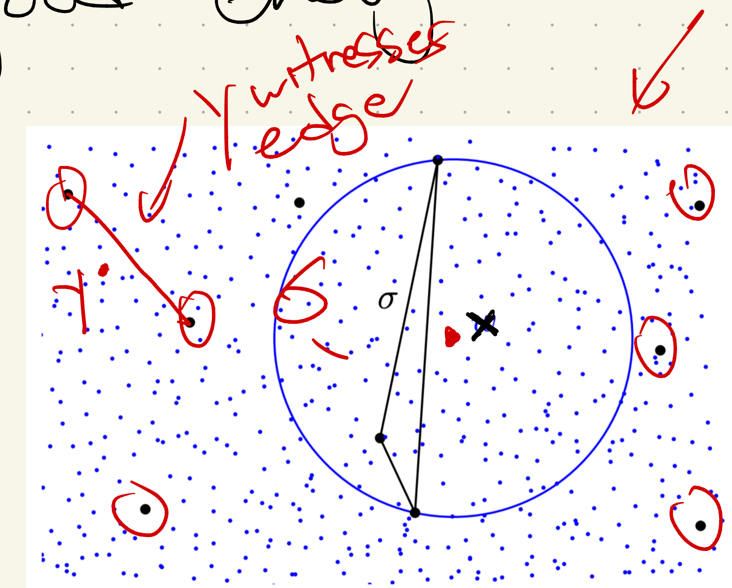
What if a point set is large?

↳ Can we find a "good enough" subsampling?

Fix 2 sets:

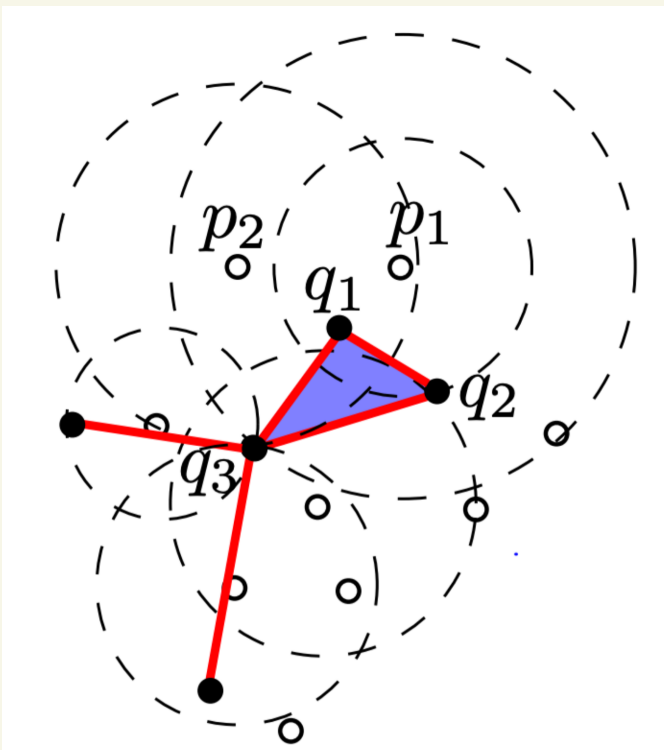
P : witnesses — big set

$Q \subseteq P$: landmarks



- A simplex $\sigma \subseteq Q$ is **weakly witnessed** by $x \in P/Q$ if $d(q,x) \le d(p,x)$ for every $q \in \sigma$ and $p \in Q \setminus \sigma$.

The witness complex $W(Q,P)$ is the collection of all σ whose faces are all weekly witnessed by a point in P/Q!



<u>Here!</u>

$q_1 q_3 \in W(P,Q)$ because p_2 weakly witnesses: $d(q_1,p_2)$ & $d(q_3,p_2)$ are closer than any other q's

$q_1 q_2 q_3 \in W(P,Q)$ because of p_1

Some facts

- If $Q \subseteq \mathbb{R}^d$,

  $\sigma \in \mathrm{Del}(Q) \iff \sigma$ is in $W(Q, \mathbb{R}^d)$

- In fact, if $Q \subseteq P \subset \mathbb{R}^d$, then

  $W(Q,P) \subseteq \mathrm{Del}(Q)$

Why care?

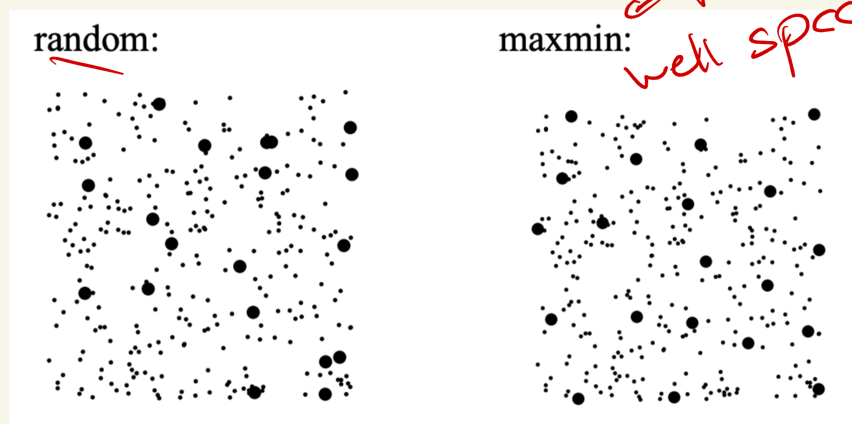
Pretty easy to compute!

The tricky part!
Usually given $P \subseteq \mathbb{R}^d$. How to pick a subset Q?

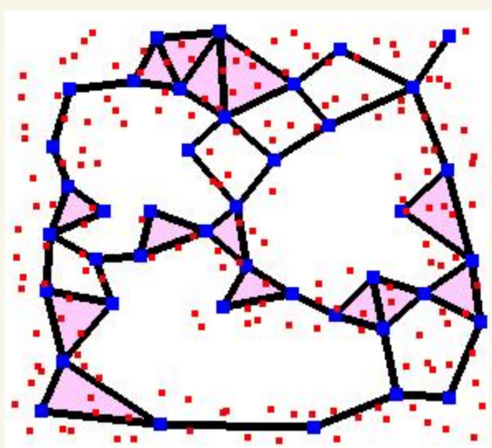
Two most common:

- Randomly
- Iteratively add furthest points

de Silva & Carlsson



← furthest: well spaced



Results vary with noise and how likely outliers are.

Guibas et al 2010

//end of sec 2.3

Homology: reminder of definitions

A field $(K, +, \cdot)$ is a set K with 2 binary operations $+$ and $\cdot$ s.t. $\forall a, b, c \in K$:

- closure: $a+b \in K$ and $a \cdot b \in K$
- Commutativity: $a+b = b+a$ and $a \cdot b = b \cdot a$ ✓
- Associativity: $(a+b)+c = a+(b+c)$ ✓
  and $a(bc) = (ab)c$
- identity: $0_K \in K$ s.t. $0_K + a = a$ ✓
  $1_K \in K$ s.t. $1_K \cdot a = a$ ✓
- inverse: $\forall a \; \exists -a$ s.t. $a+(-a) = 0_K$ ✓
  $\forall a \; \exists a^{-1}$ s.t. $a(a^{-1}) = 1_K$ ✗
- distributivity: $a(b+c) = ab+ac$

$\mathbb{Z}_2 = \{0, 1\}$

$2 \in \mathbb{Z}$

$\frac{1}{2} \notin \mathbb{Z}$

Examples: Y or N? $(\mathbb{R}, +, \cdot)$ ← yes $(\mathbb{Z}, +, \cdot)$

Vector space

A vector space over a field k is a set V with vector addition: $\forall v, w \in V$, $v+w \in V$

& scalar multiplication: $\forall a \in k$, $a\vec{v} \in V$

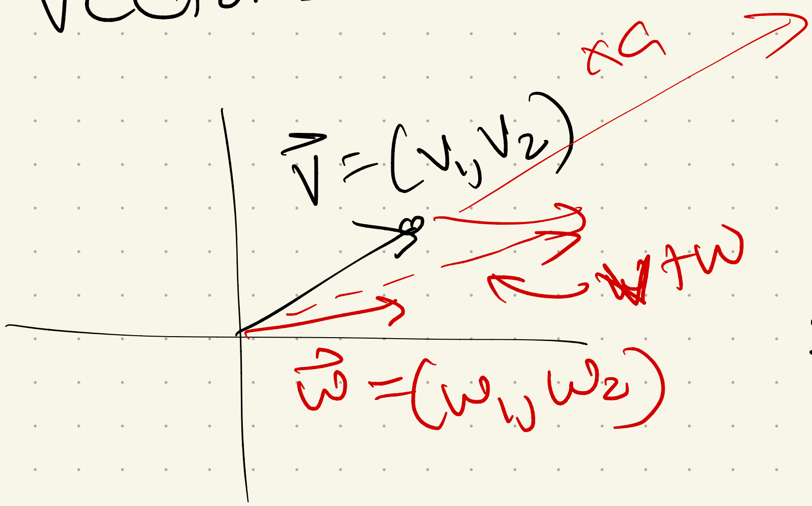
s.t. it is
- associative $(+)$: $(v+w)+x = v+(w+x)$
- commutative $(+)$: $v+w = w+v$
- identity $(+ \ \& \ \cdot)$: $\exists 0_v \in V$ & $1_k \in k$ s.t. $\forall v \in V$, $0_v + v = v$ & $1_k \cdot v = v$
- inverse $(+)$: $\forall v \in V$ $\exists w \in V$ s.t. $v+w = 0_v$
- scalar mult: $a(b\vec{v}) = (ab)\vec{v}$
- 2 kinds of distributivity:

in k $\rightsquigarrow$ $a(v+w) = av + aw$

$(a+b)w = aw + bw$

Examples:

- Vectors in $\mathbb{R}^n$:



addition: $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2)$

scalar mult: $a\vec{v}$

- Complex numbers: $x + iy$

- Function spaces $\underset{\text{set}}{\Omega} \to \underset{\text{field}}{K}$

$$(f+g)(\omega) = f(\omega) + g(\omega)$$

- Matrices & linear maps

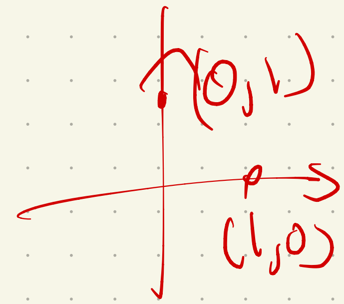
Bases

A **basis** for a vector space V is a collection of vectors $\{b_\alpha\}_{\alpha \in A}$ s.t.

- They are **linearly independent**:

  if $\sum_{\alpha \in A} c_\alpha b_\alpha = 0$ ← coefficient

  then $c_\alpha = 0$.



- They **span** V:

  $\forall v \in V$, $\exists c_\alpha \in K$ s.t. $\sum c_\alpha b_\alpha = v$

Note: All bases have the same cardinality, called the **dimension** of V.

Goal: Build a vector space from a simplical complex

Let K be a simplical complex, + fix a dimension p

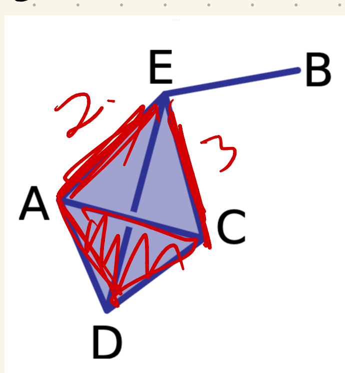
A **p-chain** is a formal sum of p-simlices, written

$$\alpha = \sum a_i \sigma_i$$

where $\sigma_i \in K$

Usually, each $a_i \in$ some field (or ring).

Example:



1 chain: $2\{a,e\} + 3\{ce\}$

2 chain: $1\{acd\}$

Adding Chains

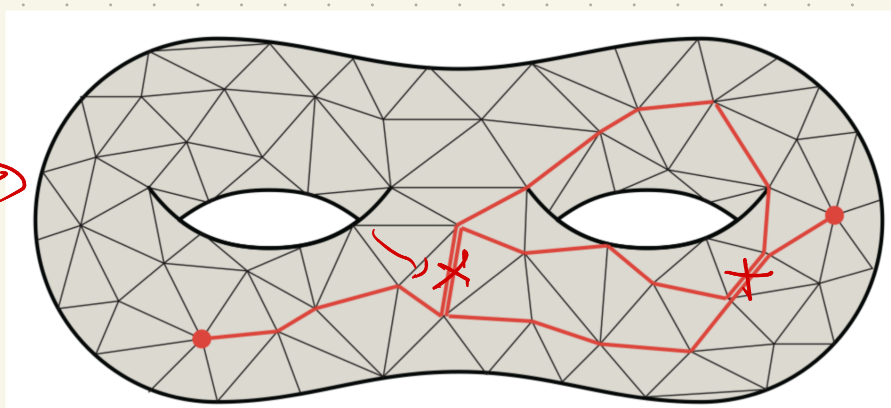
If $\alpha = \sum a_i \sigma_i$ and $\beta = \sum b_i \sigma_i$

then

$$\alpha + \beta = \sum a_i \sigma_i + \sum b_i \sigma_i$$

$$= \sum (a_i + b_i) \sigma_i$$

Example: 2-dim complex with coefficients in $\mathbb{Z}_2 = \{0, 1\}$.



1-chain:
set of cycles + paths

Chain group

The collection of p-chains with addition is called the p^{th}-chain group $C_p(K)$.

It is a vector space:

- associative $+$: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
- commutative $+$: $\alpha + \beta = \beta + \alpha$
- zero: $\vec{0} + \alpha = \alpha$ $\quad + \quad 0 = \sum 0_F \cdot \sigma_i$
- inverses: How to build $-\alpha$?

$\alpha = \sum a_i \sigma_i \quad -\alpha = \sum (-a_i)\sigma_i$

Linear Transformations

A linear transformation between 2 vector spaces V + W is a map $T: V \to W$ such that:

$\underset{\dim n}{V}$ $\underset{\dim n}{W}$

1) $T(\vec{v} + \vec{w}) =$

2) $T(a\vec{v}) =$

Representation: A matrix! Fix basis $v_1 \cdots v_n$:

$v = \sum_i a_i v_i$

$\hookrightarrow v = [\quad]$

then

$$\underbrace{\begin{pmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{pmatrix}}\underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{n \times 1} = \underbrace{\begin{pmatrix} \quad \end{pmatrix}}_{m \times 1}$$

Maps on Chain complexes

The boundary map

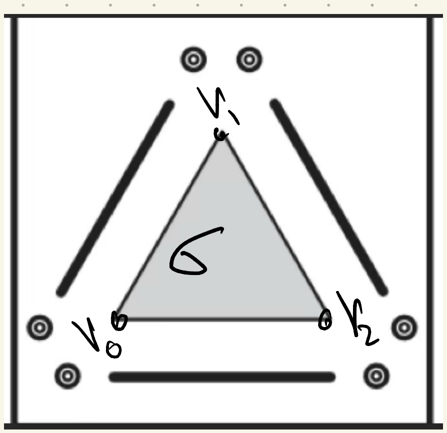
$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$$

takes $\sigma = [v_0, \dots, v_p]$

$$\longmapsto \sum_{j=0}^{p} [v_0, \dots, \hat{v}_j, \dots, v_p]$$

Here, $\hat{v}_j$ means removing simplex j.

Example:



1) $\sigma = [v_0 v_1 v_2]$

$\partial_2(\sigma) =$

2) $\partial_1([v_0 v_1] + [v_1 v_2])$

Check linearity Let $\alpha = \sum a_i \sigma_i$ and $\beta = \sum b_i \sigma_i$

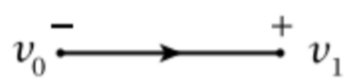
$$\partial_p(\alpha + \beta) =$$

$$= \partial_p(\alpha) + \partial_p(\beta)$$

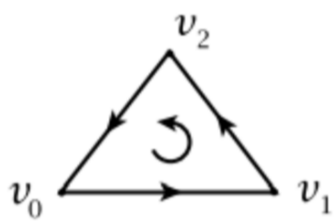
Choices of K:

Generally speaking, can study any field

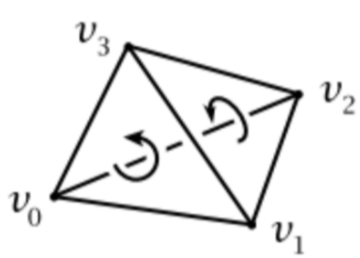
↳ or even rings!



$$\partial[v_0, v_1] = [v_1] - [v_0]$$



$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

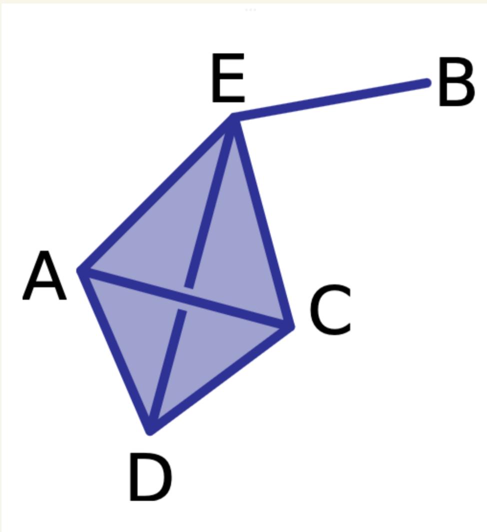


$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

But (following book), we'll focus on $\mathbb{Z}_2$.

Why?

Let's try:

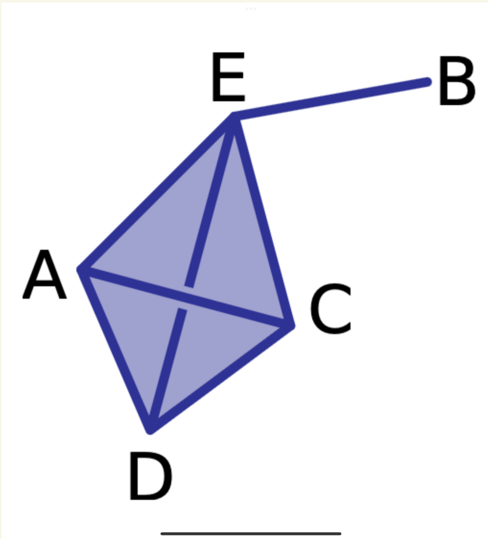


$$\partial_1([a,e]+[b,e]) =$$

$$\partial_1[[a,e]+[c,e]+[c,d]+[a,d]) =$$

$$\partial_2([ace]+[acd]) =$$

Matrix representation



$$\partial_1 : C_1(K) \to C_0(K)$$

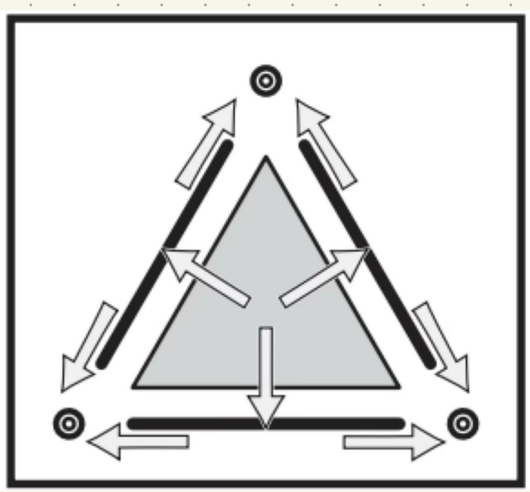
take $\alpha = \sum a_i \sigma_i$

basis?

$$\partial_1 = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Chain Complex:

$$\cdots \longrightarrow C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \to \cdots \to C_{-1} = \emptyset$$



Note: $\forall \alpha \in C_p(K)$,

$$\alpha = \sum a_i \sigma_i$$

$$\partial_{p-1} \circ \partial_p (\alpha) = 0.$$

<u>Proof</u>: For any p-simplex σ:

Cycles

Any chain in the kernal of ∂_p is called a **p-cycle.**

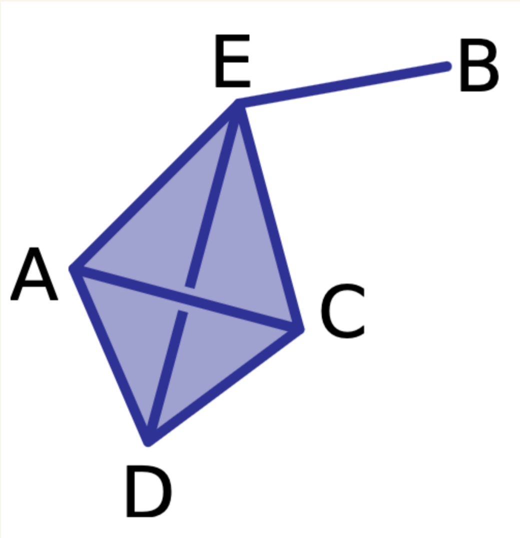
Reminder: an element x is in $\ker(f)$ if

Here: $C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$

So: a set of simplices that, after ∂_p, cancel each other out.

The set of p-cycles forms a subspace $Z_p(K) \subseteq C_p(K)$

What is a 1-cycle or 2-cycle?



Boundaries

A chain which is in the image of ∂_{p+1} is a p-boundary.

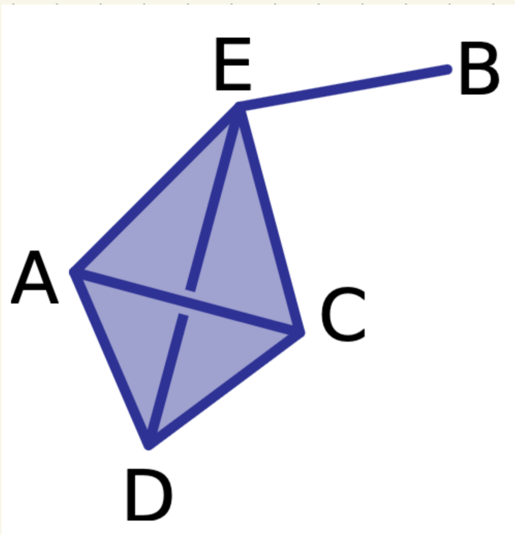
Reminder: $x \in \mathrm{im}(f)$, $f: A \to B$, if

Here: $C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K)$

& the set of p-boundaries forms a subspace $B_p(K) \subseteq C_p(K)$.

What types of things are boundaries?

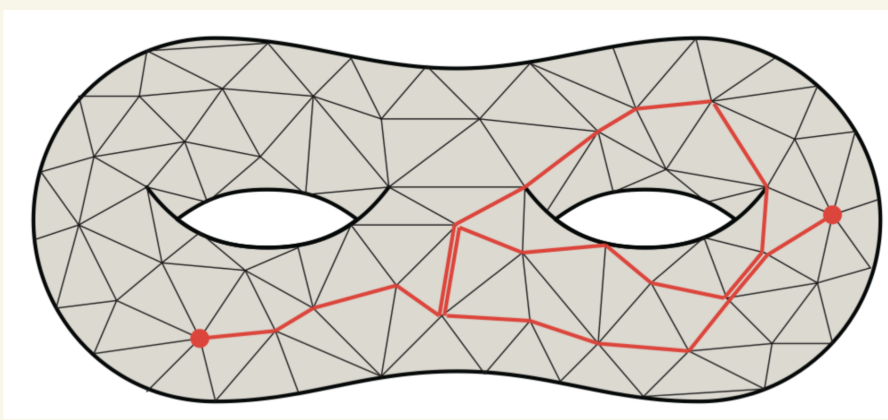
Example:



2-boundary:

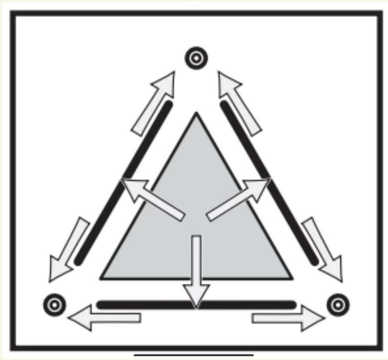
1 boundary:

Another:

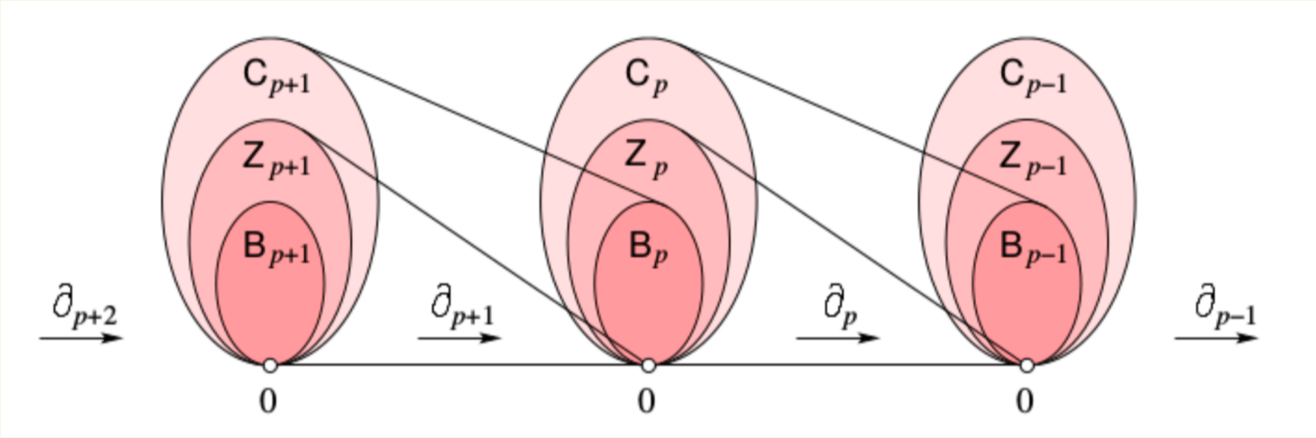


Note: Since $\partial_p \partial_{p+1}(\alpha) = 0 \ \forall \alpha \in C_{p+1}(K)$

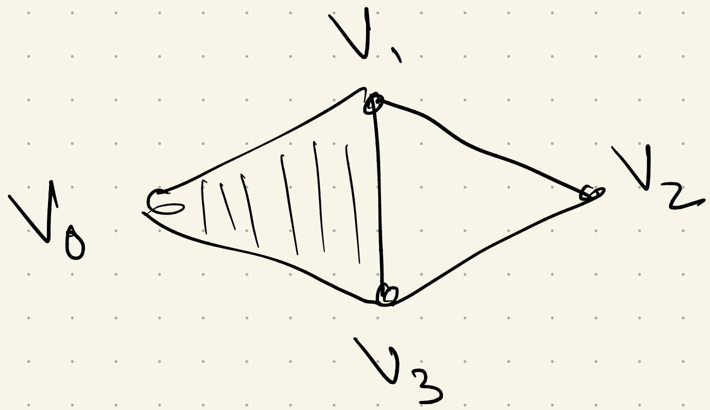
$\Rightarrow$ every p-boundary is also a p-cycle



So we get:



Example:

$K =$ 

Generators of $B_1(K)$?

Generators of $Z_1(K)$?

Quotient space

Take a vector space V over field F, and $W \subset V$ a subspace.

Define $\sim$ on V by $x \sim y$ iff $x - y \in W$.

Equivalence class of x:

$$[x] = x + W =$$

$$y \in [x] \Rightarrow$$

Then, quotient space V/W is $\{[x] \mid x \in V\}$.

Fact: V/W is a vector space with

- Scalar multiplication

  $a[x] =$

  if $y \in [x]$,

- Addition: