TDA - fall 2025

Simplicial Complexes

Recap

- Don't forget "homework 0" email. (Helping me target some later parts in class.)
- HW1: in 1 week.
  - cite sources (briefly)
  - Please type answers
  - Submit on Canvas
- Office hours: Monday 2-3pm
  Thursday 2-3pm
  (or just email / stop by!)

Correcting definition from last time

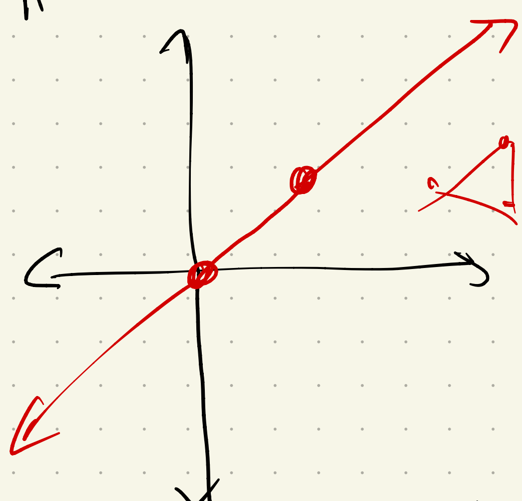
(sorry for confusion!)

Take points $a_0, \dots, a_k \in \mathbb{R}^d$, & $t_i \in \mathbb{R}$.

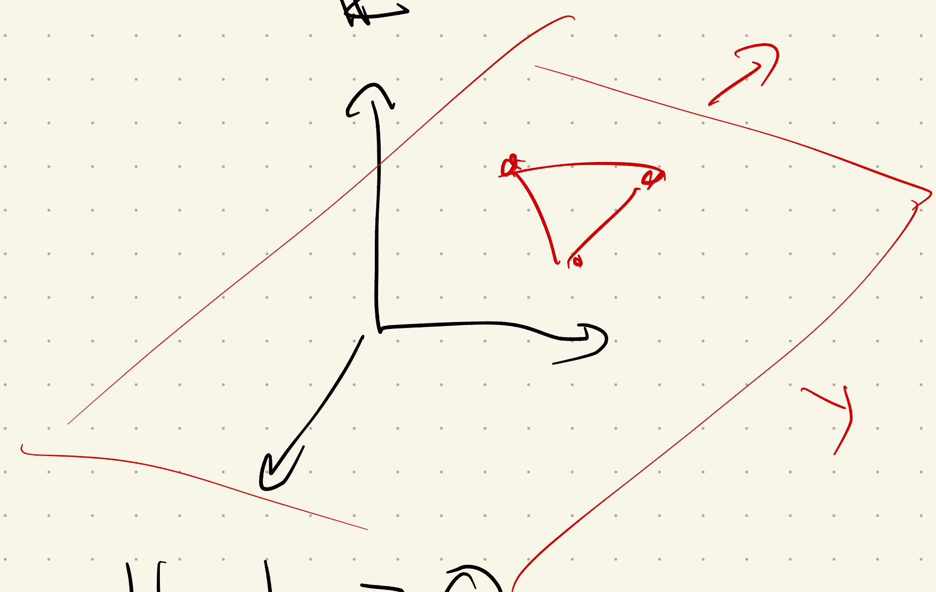
A point $x \in \mathbb{R}^d$ is an affine combination of the a_i's if $\sum_{i=0}^{k} t_i = 1$ &

$$x = \sum_{i=0}^{k} t_i a_i$$

$\mathbb{R}^2$



$\mathbb{R}^3$

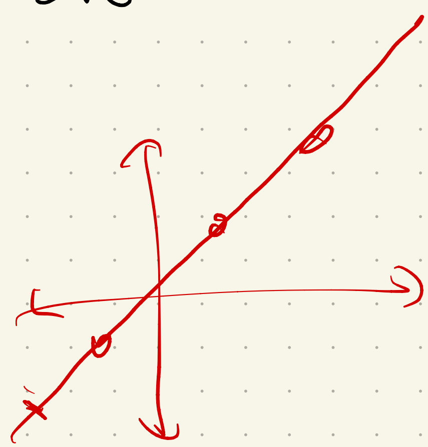


Convex combination: all $t_i \geq 0$.

The points are affinely independent if for any two combinations $x = \sum t_i a_i$ and $y = \sum u_i a_i$, $x = y \iff t_i = u_i \; \forall i$.

[Equivalently, $a_i - a_0$ vectors are linearly independent]

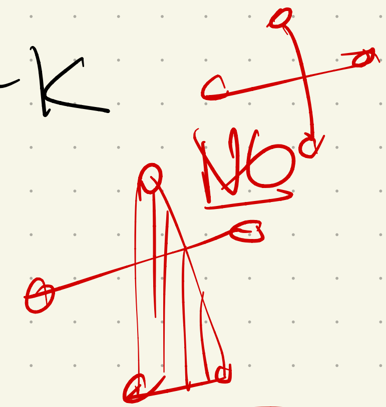
In $\mathbb{R}^d$: at most d independent vectors $\Rightarrow$ at most $d+1$ points

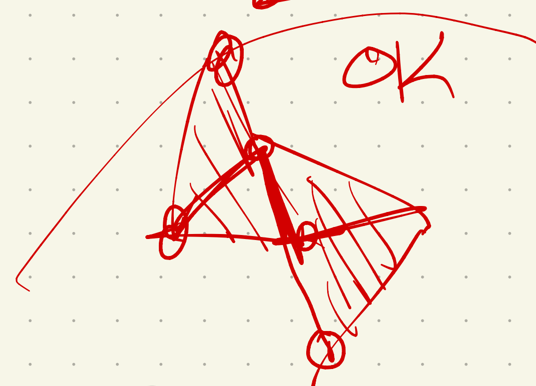


Simplicial Complex (Embedded or geometric)

A simplicial complex $K \subset \mathbb{R}^n$ is a (finite) collection of simplices in $\mathbb{R}^n$ s.t.

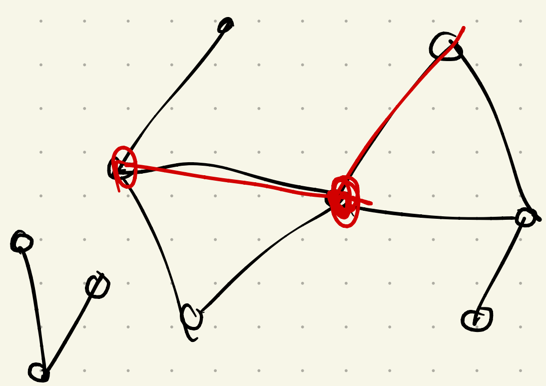
- every face of a simplex $\sigma \in K$ is also in K
- $\forall \sigma, \tau \in K$, $\sigma \cap \tau \in K$

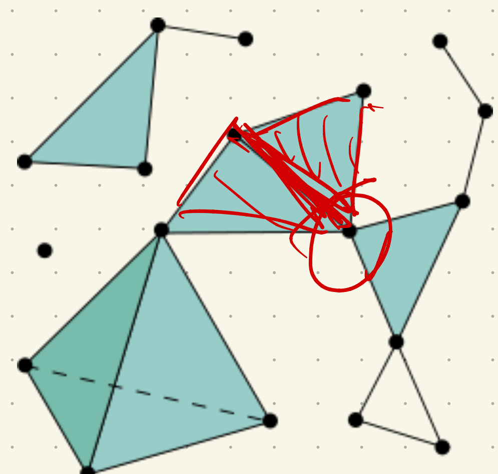




dimension of $K = \max_{\sigma \in K}\{\dim(\sigma)\}$

Examples





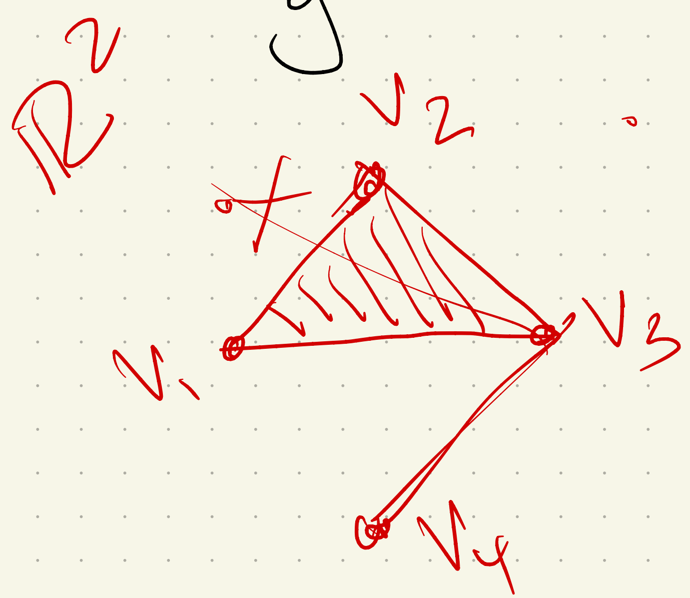
Note: Abstract simplicial complex K

a (finite) collection of (finite) non-empty subsets of a set $V = \{v_0, \dots v_n\}$ s.t.

$\sigma \in K$ and $\tau \subseteq \sigma \Rightarrow \tau \in K$

Difference:

geometric



realization – not always easy!

abstract

$\longleftarrow$

$V = \{v_1, v_2, v_3, v_4\}$

$$K = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1 v_2\}, \{v_2 v_3\}, \{v_1 v_3\}, \{v_3 v_4\}, \underbrace{\{v_1 v_2 v_3\}}_{2}\}$$

$\longrightarrow$

forget embedding

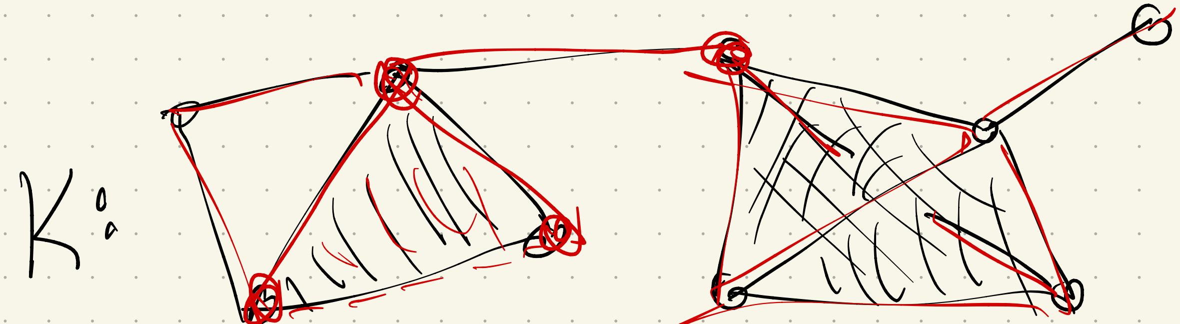
Subcomplexes & skeletons

If L is a subcollection of K that contains all faces of its elements, then L is a **subcomplex**

A subcomplex is **full** if it has all simplices from K which are spanned by vertices in L.

The subcomplex of K containing all the simplices σ with $\dim(\sigma) \leq p$ is the **p-skeleton**.

1-skeleton: graph



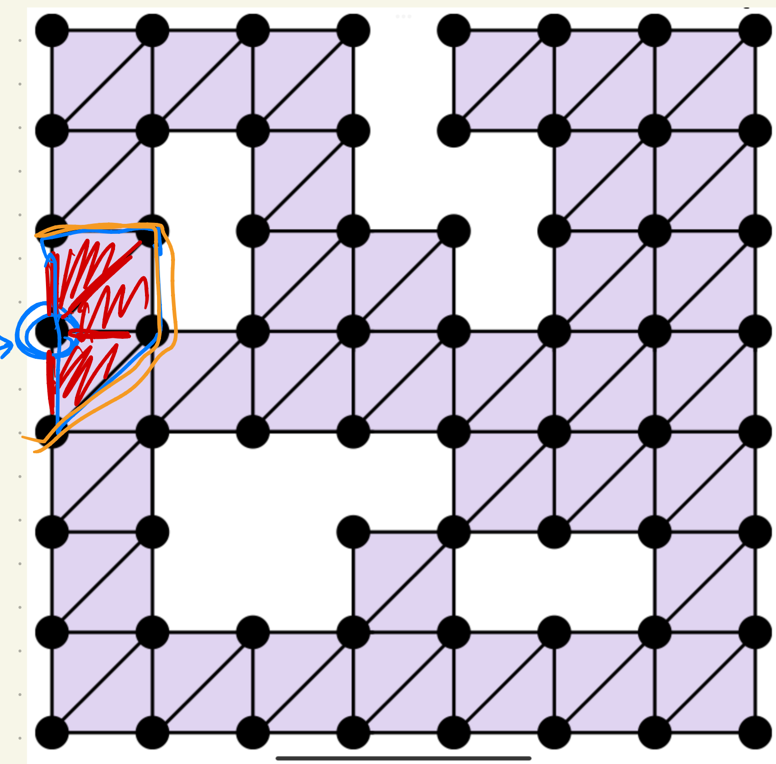
Stars & Links

The star of $\tau \in K$, $St(\tau) = \{\sigma \in K \mid \tau \leq \sigma\}$

(Warning: $st(\tau)$ is not a simplicial complex.)

$st(\tau)$

$\tau = \{v\}$



The closed star $\overline{St(\tau)}$ is the closure of $St(\tau)$.

The link of τ is $\overline{St(\tau)} - St(\tau)$

$= Lk(\tau)$

Triangulations

We say a simplicial complex K is a **triangulation** of a manifold M if the underlying space $|K|$ is homeomorphic to M.

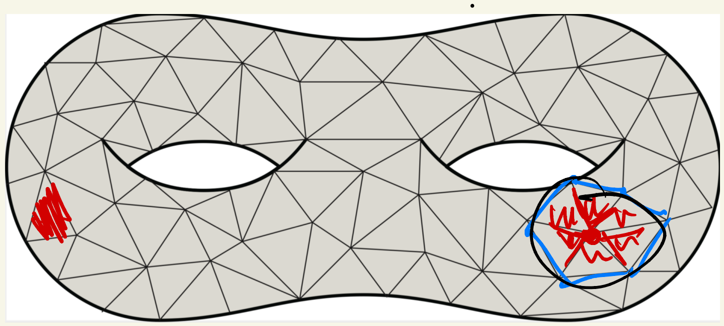
Note: If M is a k-manifold, $\dim(K)$ must be k also.

Useful facts:

$$\forall v \in K, \quad |St(v)| \simeq \mathbb{B}^k_0 \text{ or } \mathbb{H}^k_0$$

$$\text{and} \quad |Lk(v)| \simeq \mathbb{S}^{k-1} \text{ or } \mathbb{B}^{k-1}_0$$

Example.

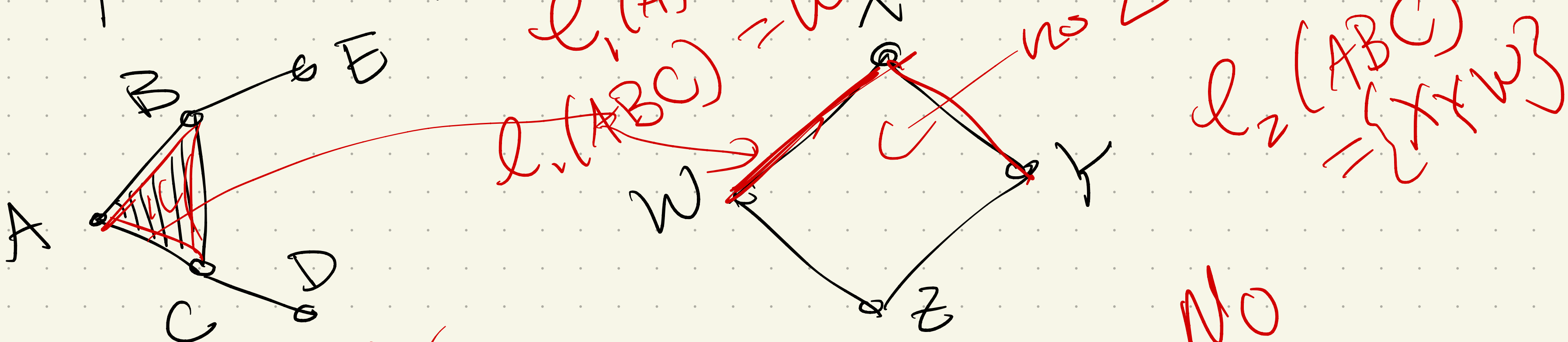


dim = 2

Simplicial maps

A map $f: K_1 \to K_2$ is called simplicial if $\forall\, \tau = \{v_0, \dots, v_k\} \in K_1$, we have the simplex $f(\tau) = \{f(v_0), \dots, f(v_k)\} \in K_2$

Example: Simplicial?



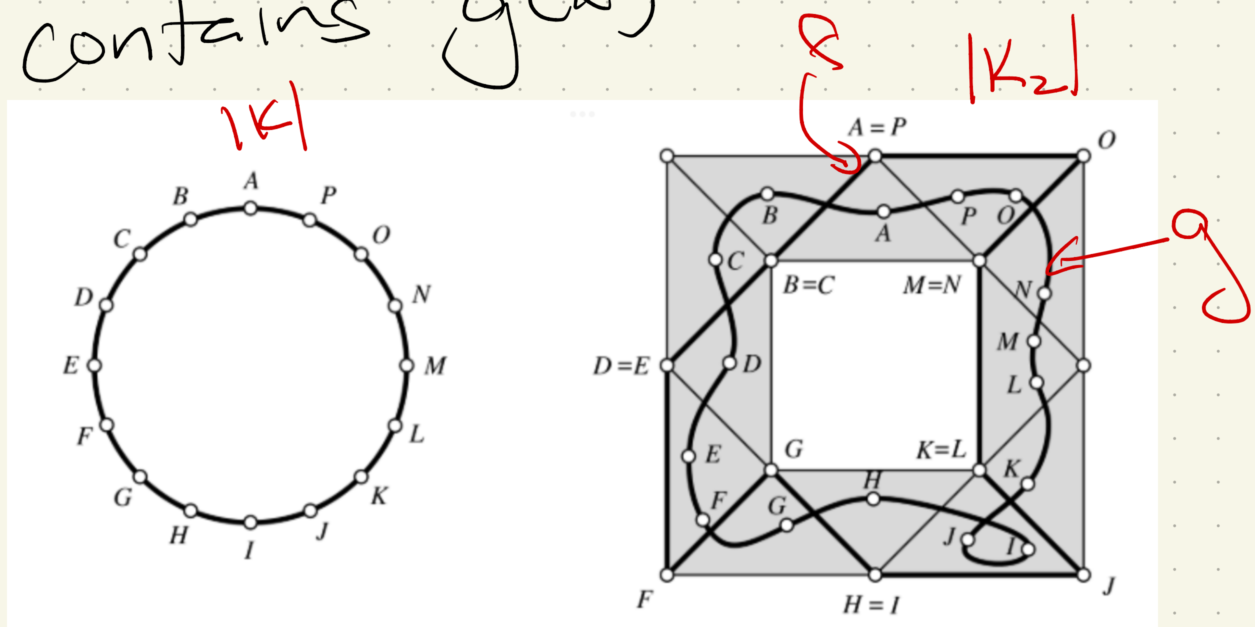
ℓ_1: $A \mapsto W$, $B \mapsto X$, $C \mapsto X$, $D \mapsto Y$, $E \mapsto Y$ — Yes

ℓ_2: $A \mapsto X$, $B \mapsto Y$, $C \mapsto W$, $D \mapsto Z$, $E \mapsto Z$ — No

Fact: Every continuous function $g: |K_1| \to |K_2|$ can be "approximated" by a simplicial map f on appropriate subdivisions of K_1 & K_2.

Here: for a point $x \in |K_1|$, $f(x)$ belongs to the minimal closed simplex $\sigma \in L_2$ that contains $g(x)$

Two maps shown: continuous g & simplicial f



Point clouds

Let X be a finite point set in a metric space (M, d).

↳ often $(\mathbb{R}^d, \ell_2)$

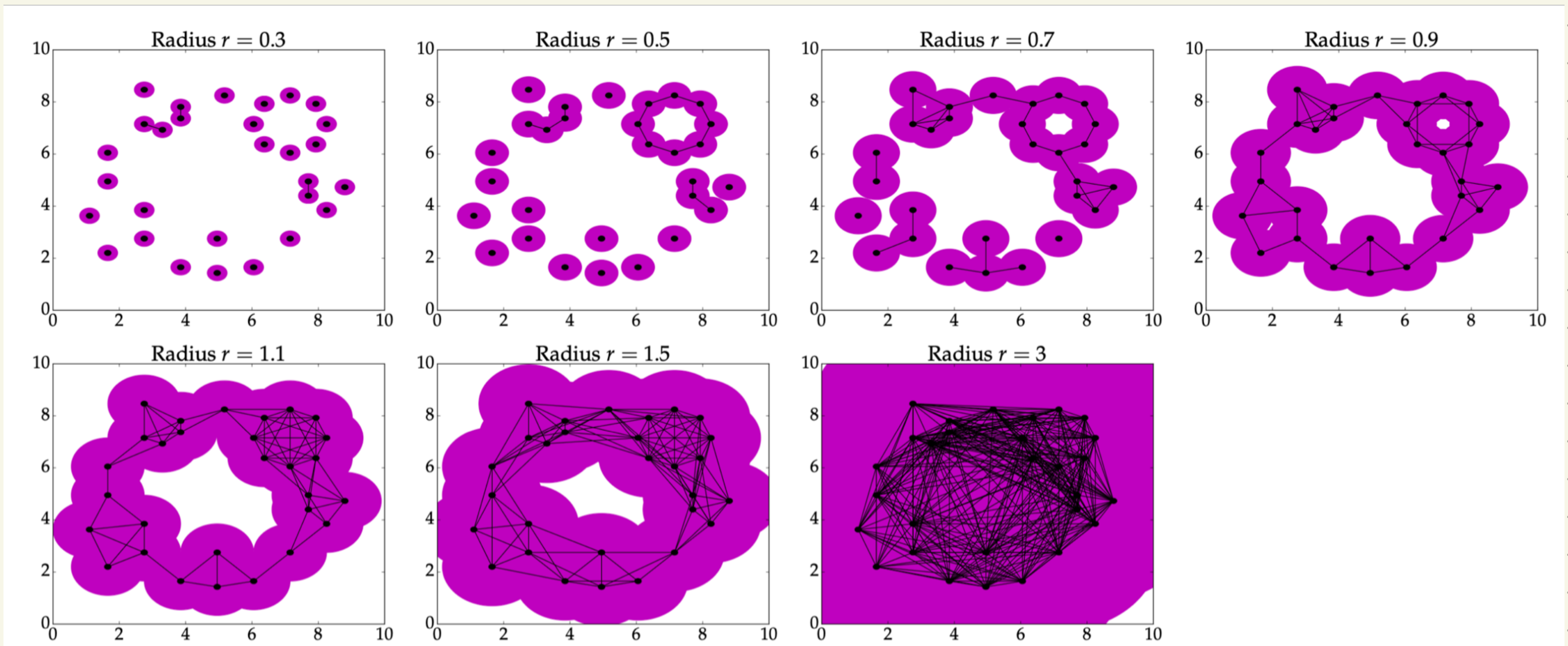
Note: topology is pretty boring!

$|X|$

Let $B(x,r) = \{ y \in M \mid d(x,y) \leq r \}$
(so these are closed)

Goal: Study how these balls interact.

$\mathbb{R}^2$

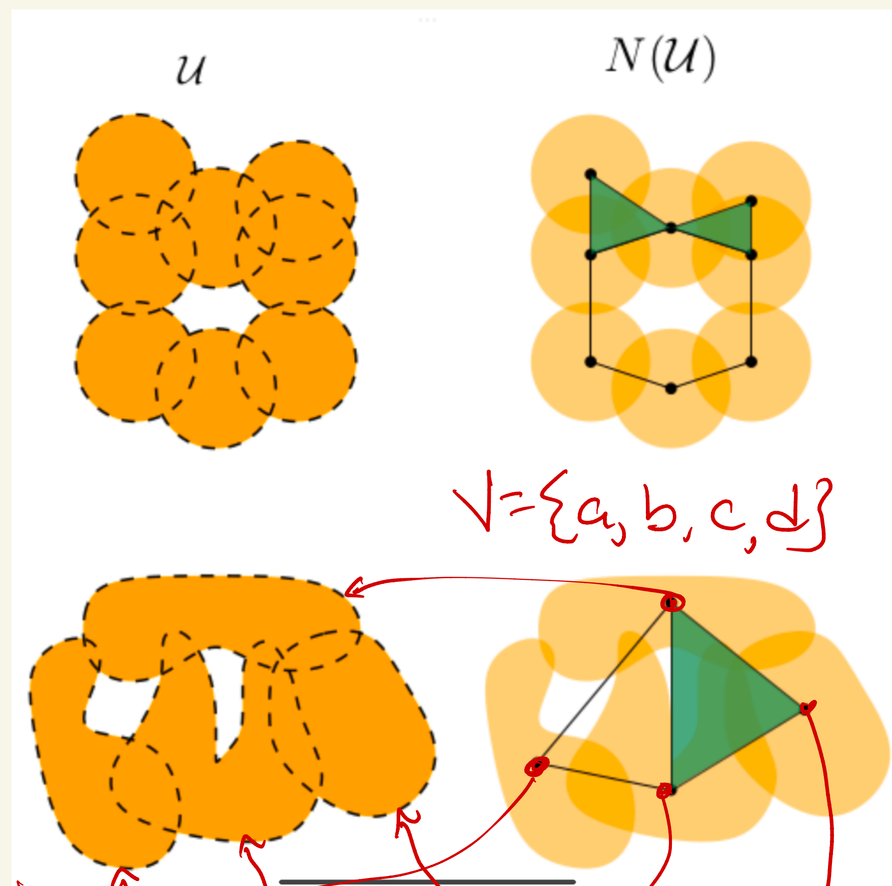


Note: there isn't a single correct r!

Given a finite collection of sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, the nerve of $\mathcal{U}$, $N(\mathcal{U})$, is the simplicial complex with vertex set A, where $\{\alpha_0, \ldots, \alpha_k\} \subseteq A$ is a k-simplex $\in N(\mathcal{U})$

$$\Longleftrightarrow$$

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha(k)} \neq \emptyset$$



3 sets intersecting $\hookrightarrow \emptyset$

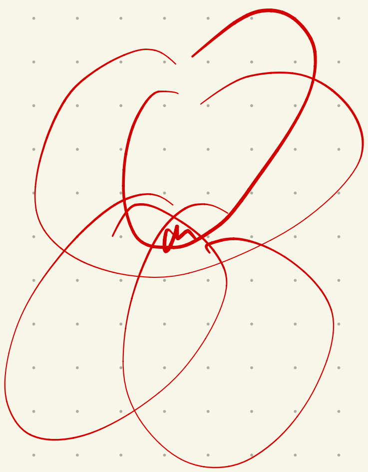
$\{ab\} \in N(\mathcal{U})$?

Check: This is an abstract simplicial complex.

Need if $\sigma \in K$ + $\tau \leq \sigma$ $\Rightarrow \tau \in K$

Here: if $\sigma = \{\alpha_0, \dots, \alpha_k\}$

$\Rightarrow U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \neq \emptyset$



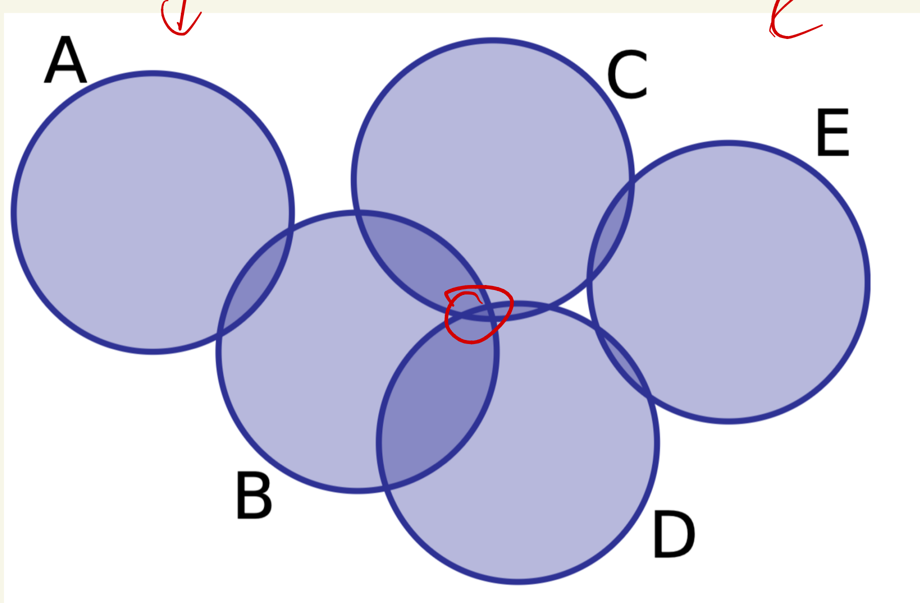
why does any $\tau \leq \sigma$ also appear?

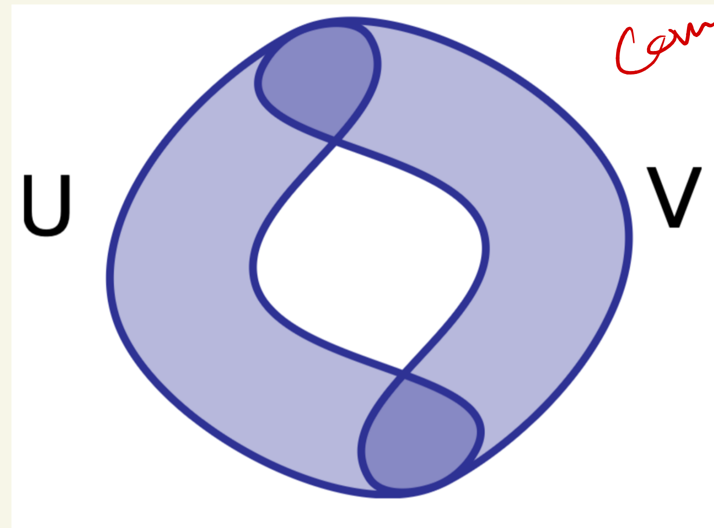
any sub-$\cap$ also has $\neq \emptyset$

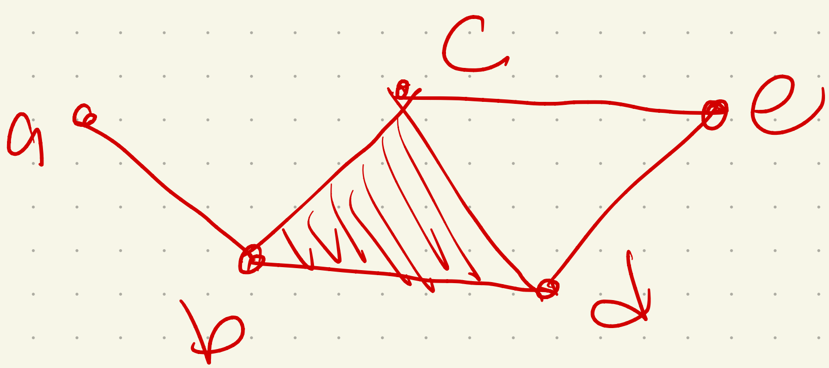
Some examples to try:

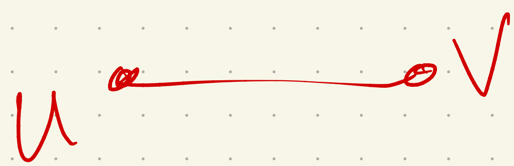
α's

$U \cap V$ has 2 components









Difference: balls somehow reflect topology

Nerve Lemma

Given a finite cover $\mathcal{U}$ (open or closed) of a metric space M, the underlying space $|N(\mathcal{U})|$ is homotopy equivalent to M if every non-empty intersection $\bigcap_{i=0}^{k} U_{\alpha_i}$ of cover elements is homotopic to a point (i.e. is contractible).

↳ balls work, not necessary

Why we care: If cover has contractible intersections, the nerve is a good proxy for understanding M.

<u>But</u>: lots of ways to take open sets around points!
(And lots where intersections are contractible.)

We'll focus on several popular ones:

metric spaces
- Čech complex
- Vietoris-Rips complex

$\mathbb{R}^d$
- Delaunay complex
- Alpha complex

Čech complex:

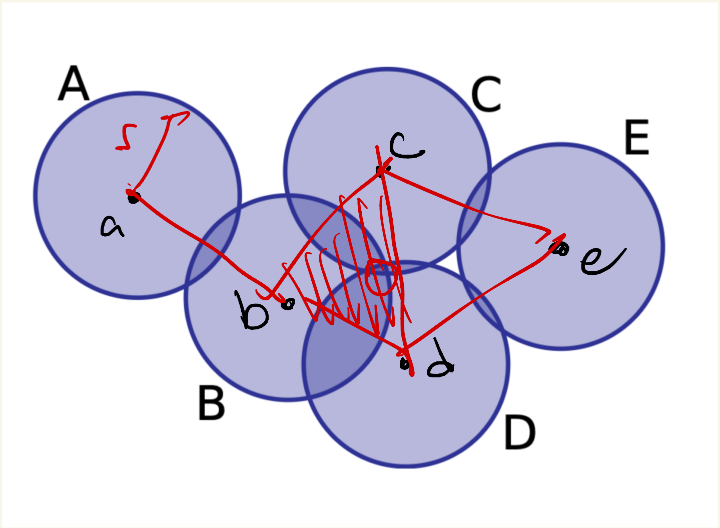
Let $P \subset (M, d)$ be a finite point cloud, & fix some $r > 0$. The Čech complex is

$$C^r(P) := \{\sigma \subseteq P \mid \bigcap_{x \in \sigma} B(x,r) \neq \emptyset\}$$

$$= N(\{B(x,r)\}_{x \in P})$$

Example:

5 points in $\mathbb{R}^d$, $d = \ell_2$



$V = \{a, b, c, d, e\}$

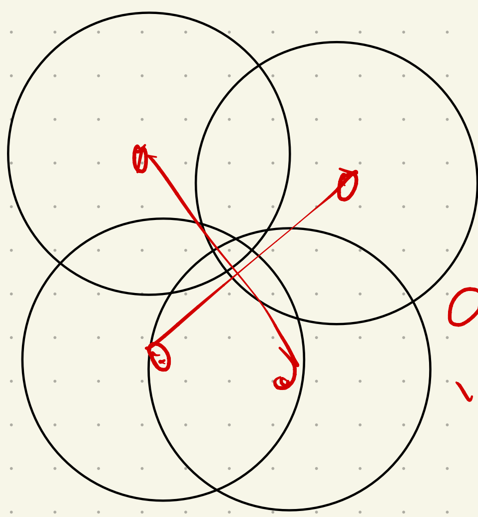
$\{a\}, \{ab\}, \ldots$

Warning: $C^r(P)$ is an abstract simplicial complex!

The "obvious" map into $\mathbb{R}^d$ does not always get you a geometric complex.

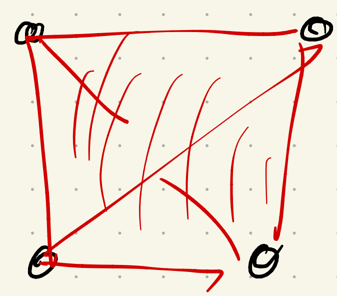
What breaks?

4 points in $\mathbb{R}^2$:



all intersect

$\longrightarrow$ abstract complex



simplices can $\cap$ in non simplicial

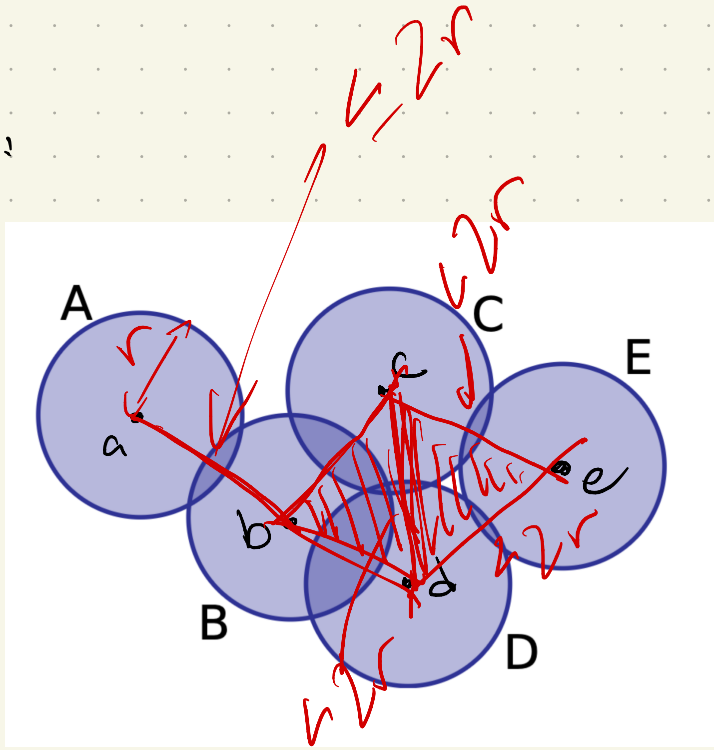
? take centers

Rips Complex

Given $P \subset (M, d)$ a finite point set in a metric space, & $r > 0$,

$$VR^r(P) := \{\sigma \subseteq P \mid d(p,q) \leq 2r \ \forall p,q \in \sigma\}$$

Example:



Relationship

Fact: The Rips complex is completely determined by its 1-skeleton.

Why care? $\rightsquigarrow$ computation!

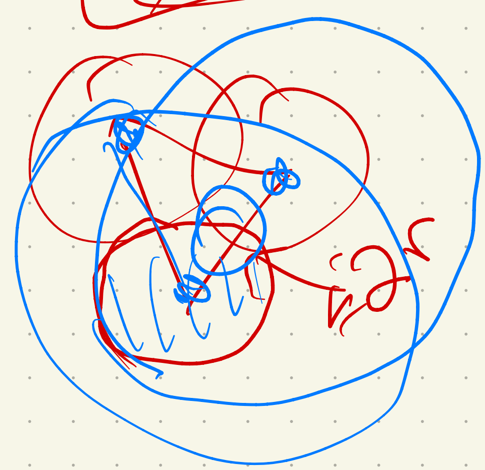
Rips-Čech lemma

Given a point cloud $P \subset (M, d)$.

$\forall r > 0$, $\quad C^r(P) \subseteq \boxed{VR^r(P)} \subseteq C^{2r}(P)$

Proof:

geometry:



Voronoi diagrams

Given a set of points P in $\mathbb{R}^d$, the Voronoi cell for site $p \in P$ is

$$V_p = \{x \in \mathbb{R}^d \mid d(x,p) \leq d(x,q) \; \forall q \in P\}$$

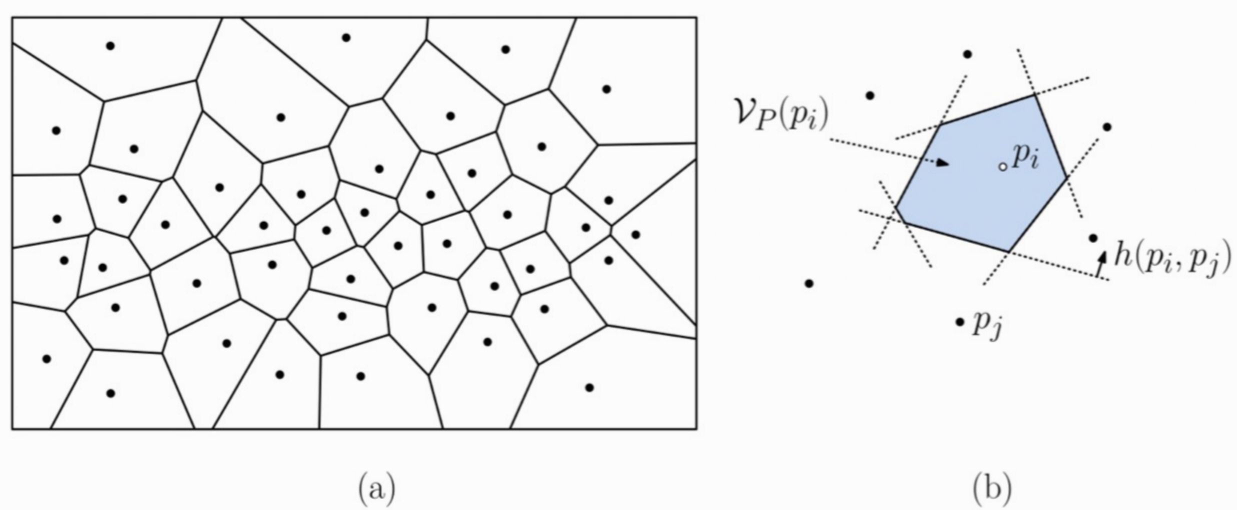


Fig. 55: Voronoi diagram Vor(P) of a set of sites.

This tesselates $\mathbb{R}^d$, & the collection of cells is the Voronoi diagram $\text{Vor}(P) = \{V_u \mid u \in P\}$

Why?

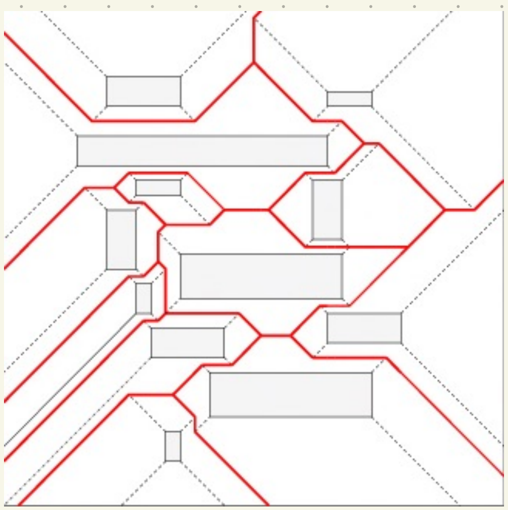
Super useful!

- Closest point queries
- Shape analysis
- Clustering

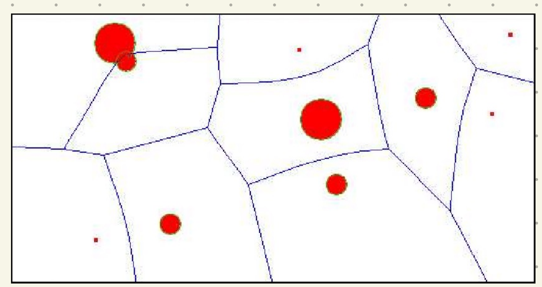
$\Rightarrow$ $\boxed{\text{lots of code}}$

Even variants for other metrics on $\mathbb{R}^d$!

ℓ_1 distance, polygons



weighted Voronoi

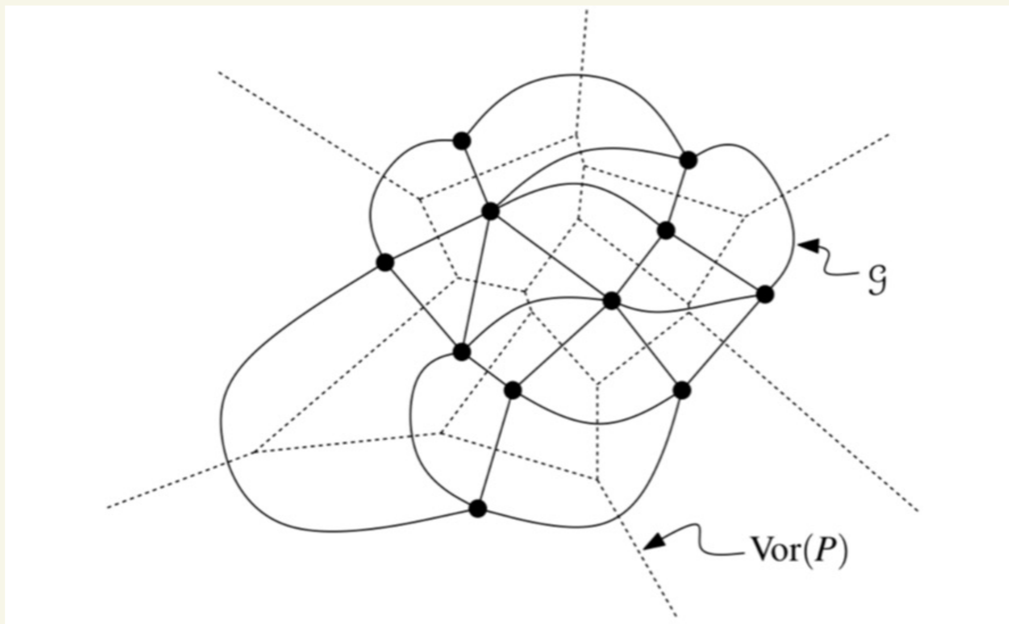


Why we care

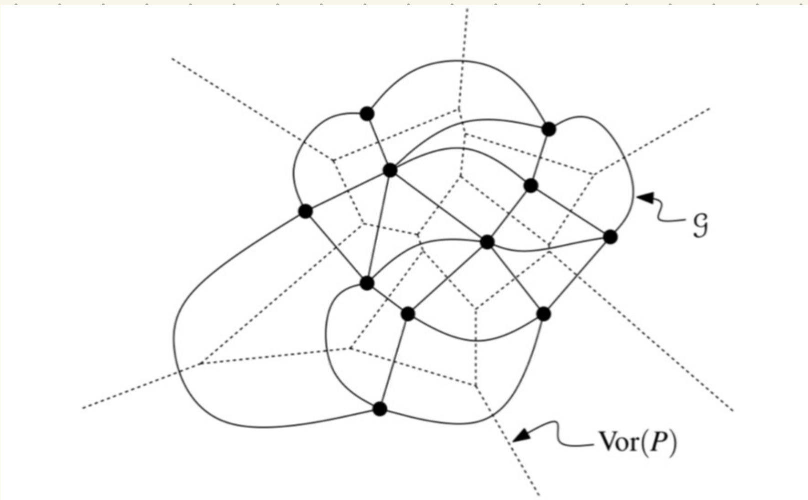
The **Delaunay complex** of $P \subseteq \mathbb{R}^d$ is the nerve of the Voronoi diagram!

$$\mathrm{Del}(P) = \left\{ \sigma \subseteq P \;\middle|\; \bigcap_{u \in \sigma} V_u \neq \emptyset \right\}$$

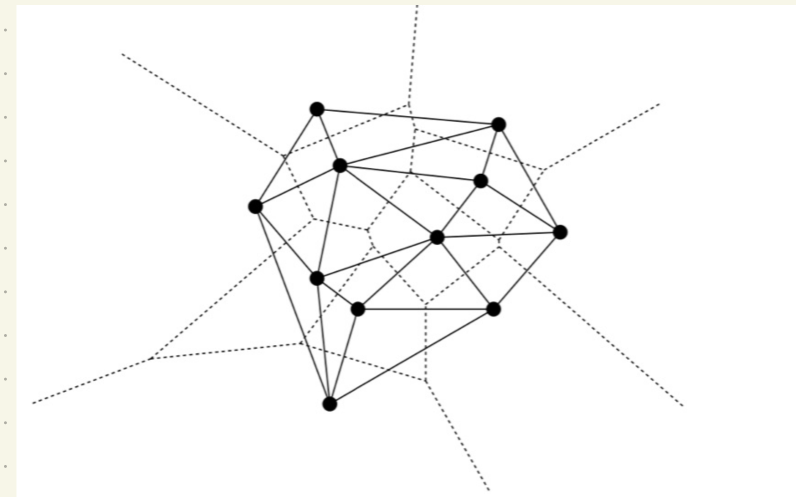
Note: Still an abstract simplicial complex!



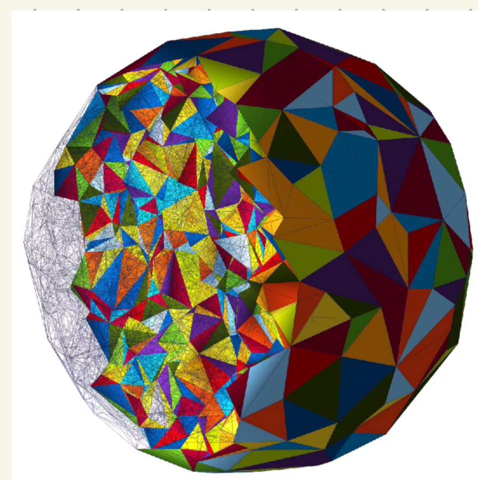
Fact: The "obvious" embedding of Del(P) gives a geometric simplicial complex:



$\longrightarrow$



Note: no parameter r here – Del(P) & Vor(P) are fixed.



Why is it nice?

A triangulation of a point set $P \subset \mathbb{R}^d$ is a geometric simplicial complex with point set P whose simplices tesselate the convex hull of P.

Among all triangulations, Del(P):

1) minimizes the largest circumcircle for $\triangle$'s in the complex (in $\mathbb{R}^2$)

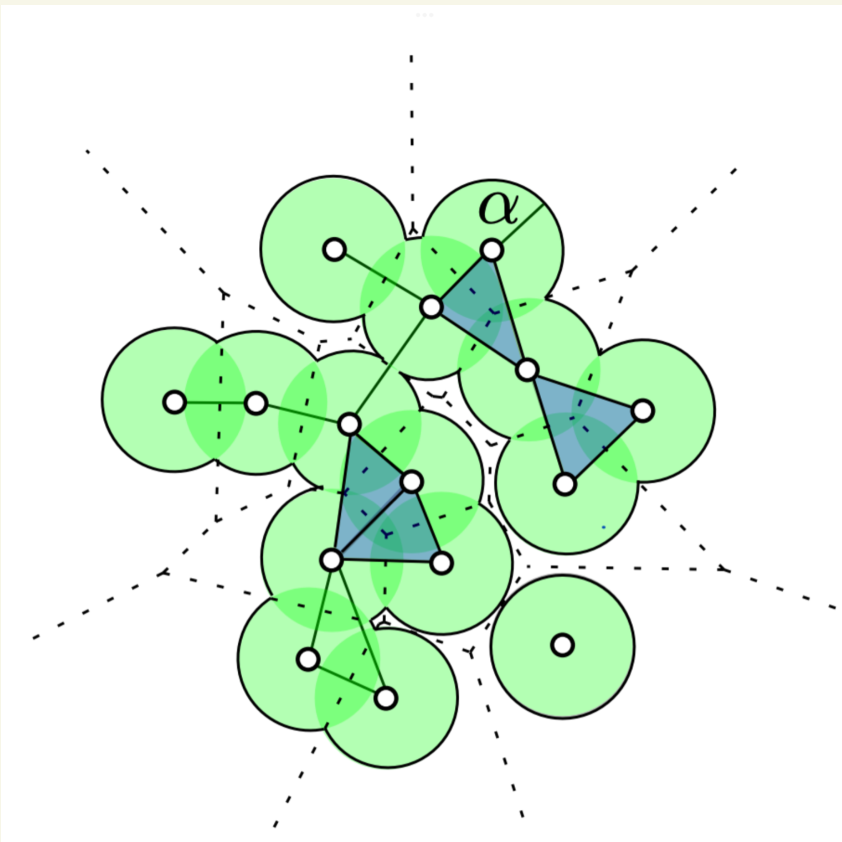
2) maximizes the minimum angle of $\triangle$'s in the complex (in $\mathbb{R}^2$)

3) All minimum enclosing balls of simplices are empty, & largest is minimized

Adding r back in:

Let $D_p^\alpha := \{ x \in B(p, \alpha) \mid d(x,p) \le d(x,q) \ \forall q \in P \}$

$$= B(p, r) \cap V_p$$

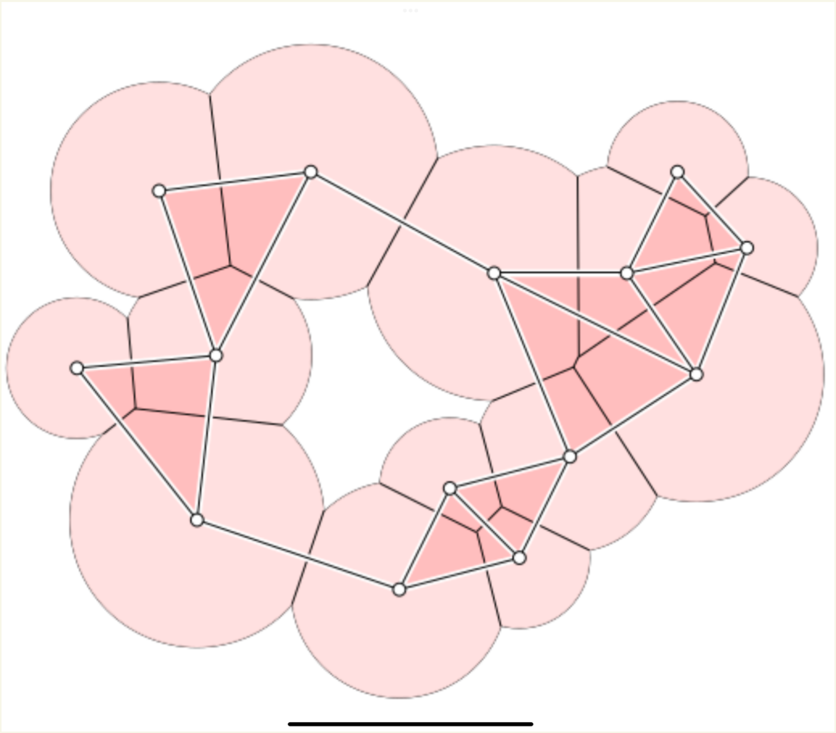


The alpha complex

$$\mathrm{Del}^\alpha(P) = N\left(\{ D_p^\alpha \mid p \in P \}\right)$$

Properties

- $Del^{\alpha}(P) \subseteq Del(P)$
- $Del^{\alpha}(P) \subseteq \check{C}(r)$
- $Del^{\alpha}(P)$ has the same homotopy type as the union of balls of radius r.



The book covers 2 other types of complexes: witness complex & graph induced complex.

Both describe ways to "sparsify" data:

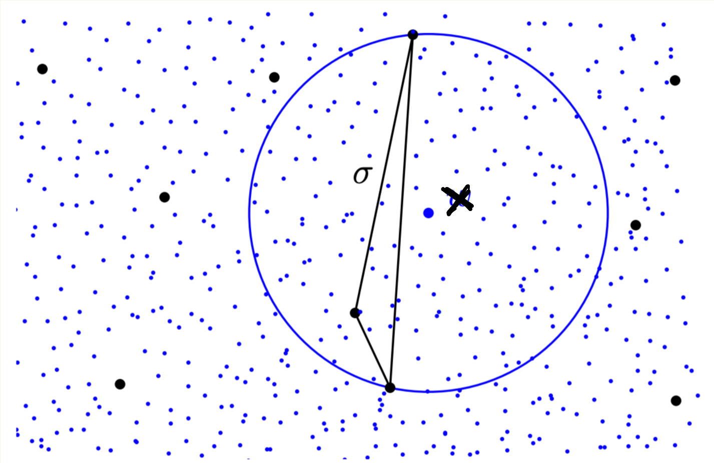
Find a "good enough" subsampling of a point set P:

take $Q \subseteq P$ & define a simplicial complex on Q (but using P to build simplices)

Witness Complex

What if a point set is large?

↳ Can we find a "good enough" subsampling?



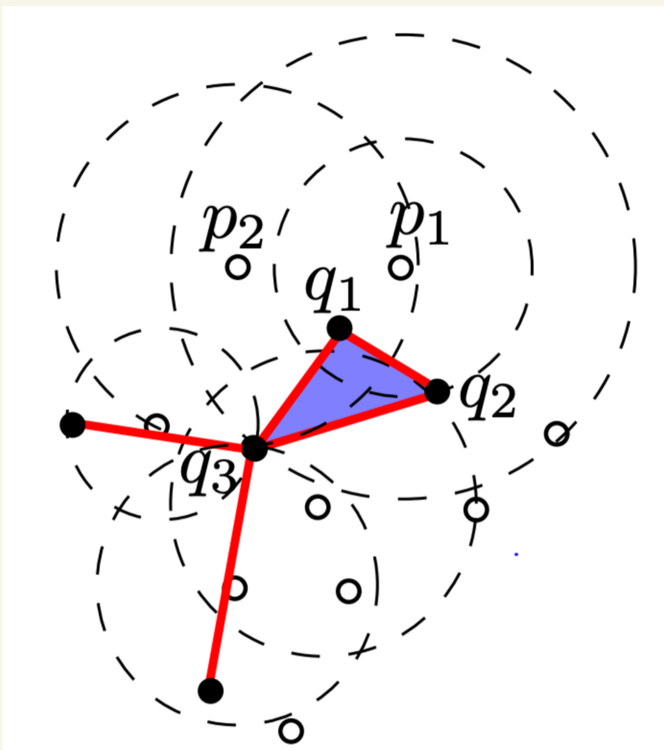
Fix 2 sets:

P : witnesses

$Q \subseteq P$: landmarks

- A simplex $\sigma \subseteq Q$ is **weakly witnessed** by $x \in P/Q$ if $d(q,x) \leq d(p,x)$ for every $q \in \sigma$ and $p \in Q \setminus \sigma$.

The witness complex $W(Q,P)$ is the collection of all σ whose faces are all weakly witnessed by a point in P/Q:



Here:

$q_1 q_3 \in W(P,Q)$ because p_2 weakly witnesses: $d(q_1,p_2)$ & $d(q_3,p_2)$ are closer than any other q's

$q_1 q_2 q_3 \in W(P,Q)$ because of p_1

Some facts

- If $Q \subseteq \mathbb{R}^d$,

  $\sigma \in Del(Q) \iff \sigma$ is in $W(Q, \mathbb{R}^d)$

- In fact, if $Q \subseteq P \subset \mathbb{R}^d$, then

  $W(Q, P) \subseteq Del(Q)$

Why care?

Pretty easy to compute!

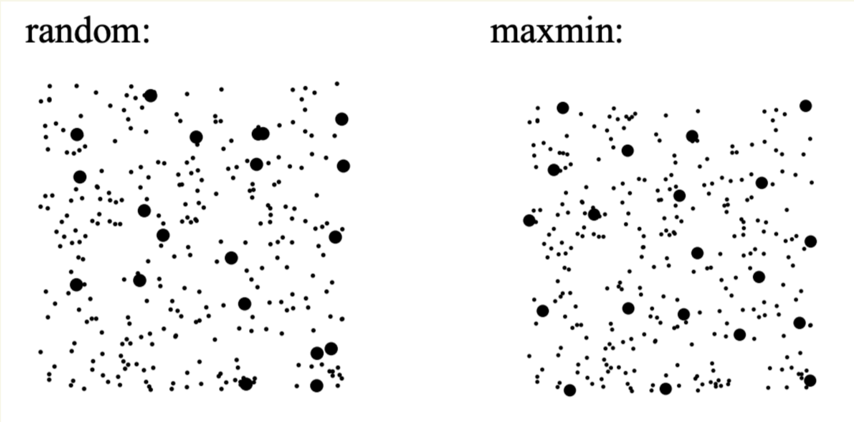
The tricky part!

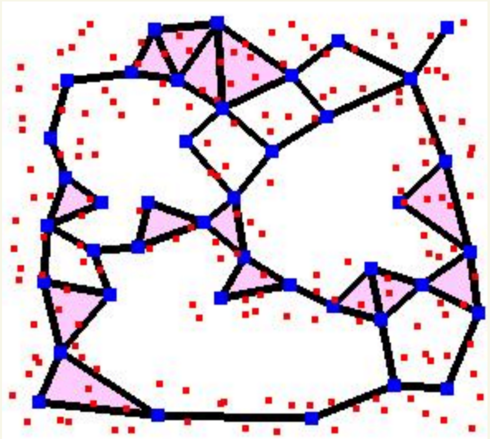
Usually given $P \subseteq \mathbb{R}^d$. How to pick a subset Q?

deSilva & Carlsson

Two most common:

- Randomly
- Iteratively add furthest points





Results vary with noise and how likely outliers are.

Guibas et al 2010