

TDA - fall 2025

Extended
Persistence (cont)
+ Reeb graphs



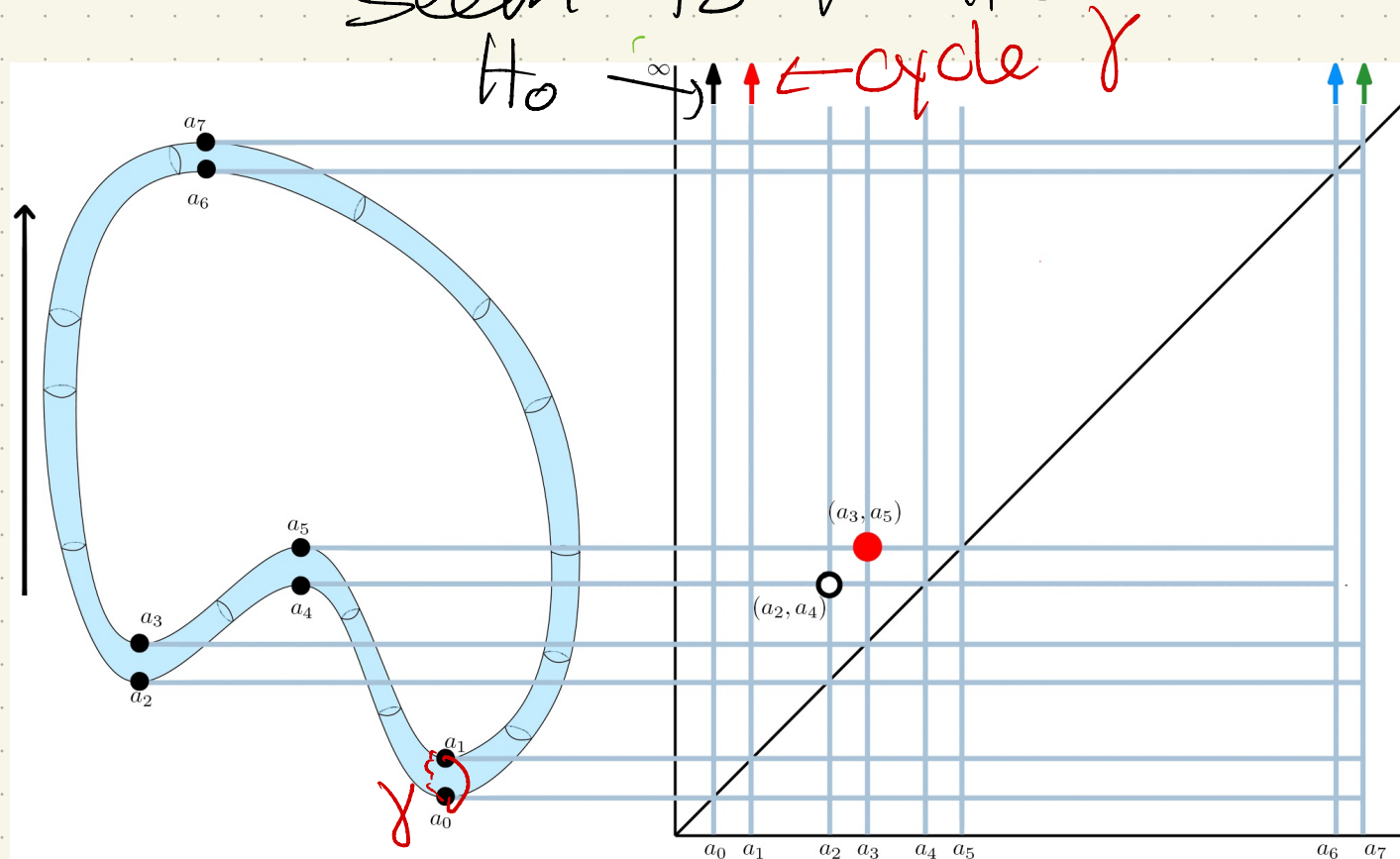
Recap

- Paper chase: due October 17
(sorry for typo on webpage!)
- Next assignment will be final
project proposal
- Shorten office hours today

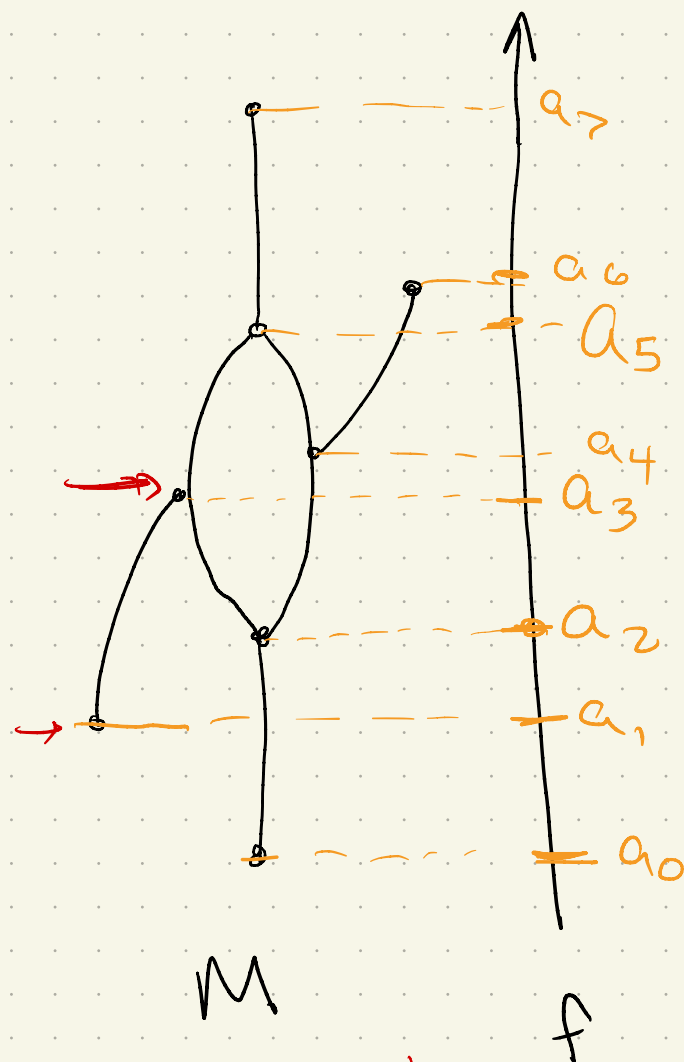
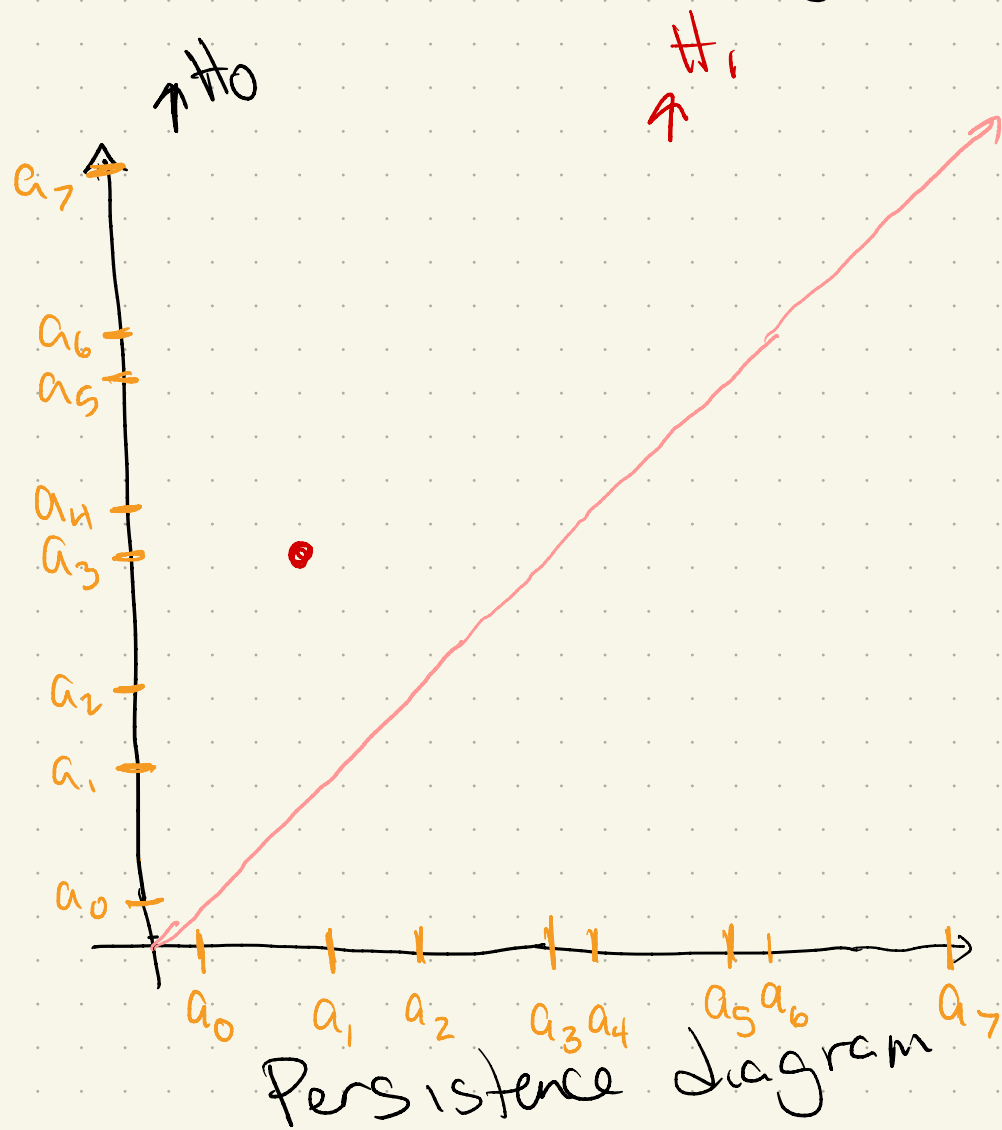
Extended Persistence

Odd parts of persistence:

- points at infinity
- some Morse critical points don't seem to matter



Really becomes obvious on Reeb graphs
(more next week)



what about the rest!

Agarwal-Edelsbrunner-Harer-Wang 2006

⇒ Cohen-Steiner, Edelsbrunner, Harer 2009

Use relative homology to find other critical points, + get better pairing.

Relative homology Fix $L \subseteq K$

Define $C_p(K, L) := C_p(K) / C_p(L)$

$$+ [\alpha] = \{ \gamma \in C_p(K) \mid \alpha + \gamma \in C_p(L) \}$$

Then maps extend to homology.

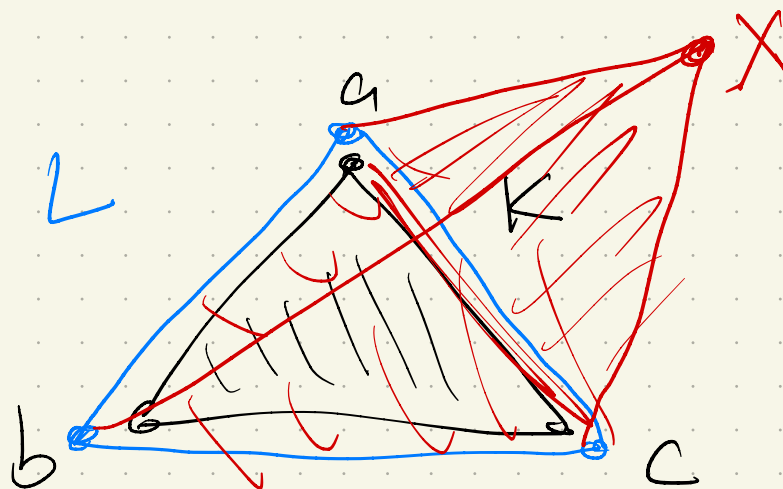
Remember a month ago? →

Fun fact

$$\text{Let } K^* = K \cup \{x\} \cup \{\underline{\sigma} \cup \{x\} \mid \underline{\sigma} \in L\}$$

"coned off"

tetrahedra



Theorem:

$$H_p(K, L) = H_p(K^*) \text{ for } p > 0$$

$$\& \beta_0(K, L) = \beta_0(K^*) - 1$$

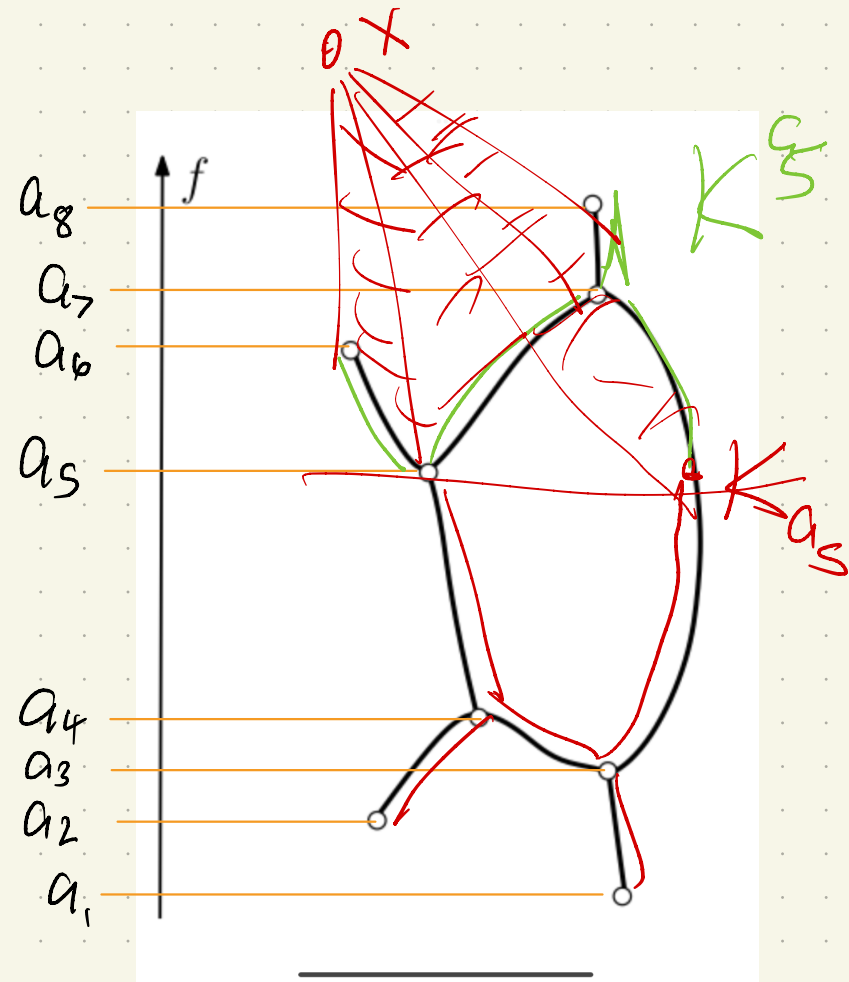
Here, we want to look at relative homology of the superlevel sets:

Given $f: K \rightarrow \mathbb{R}$

$$K_a = \{ \sigma \in K \mid f(\sigma) \leq a \}$$

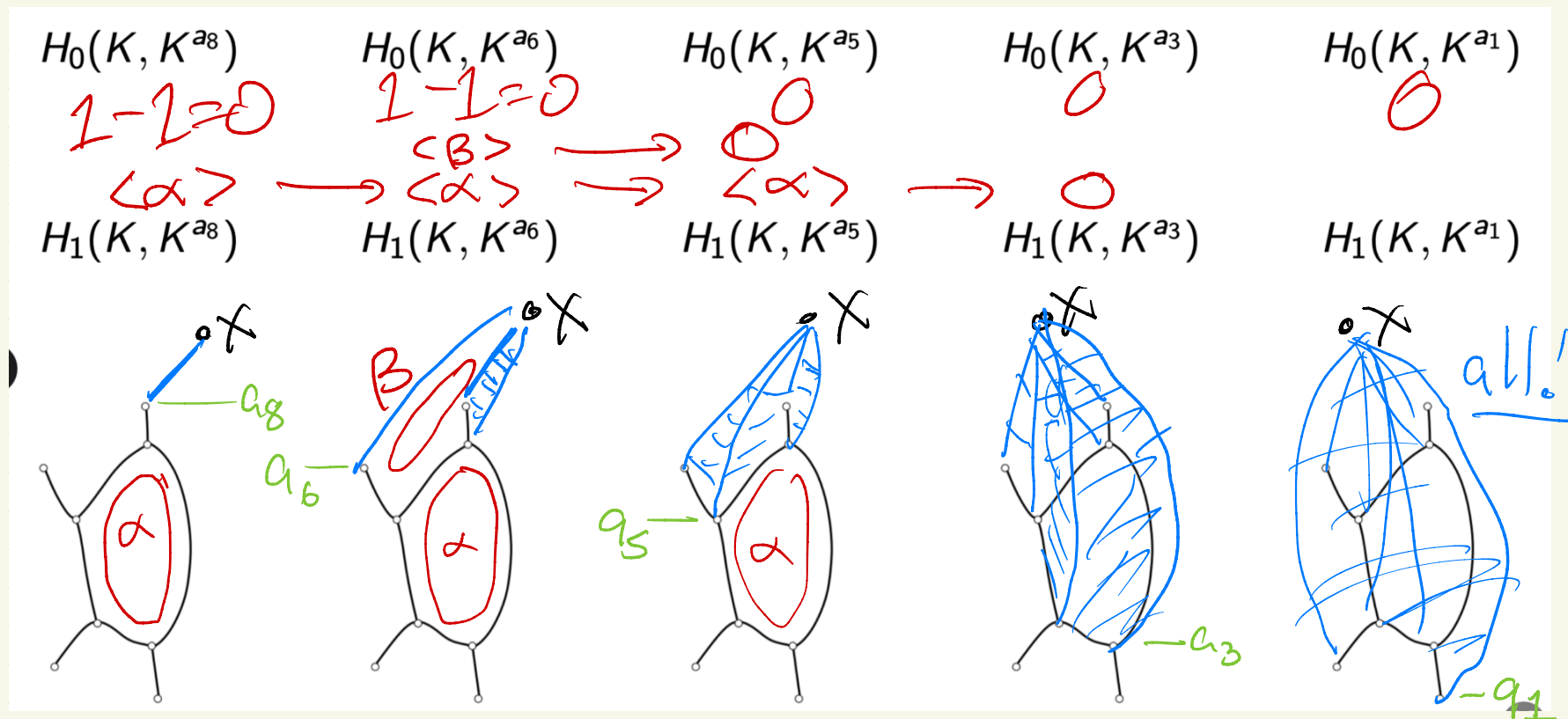
$$K^a = \{ \sigma \in K \mid f(\sigma) \geq a \}$$

+ study $H_p(K, K^a)$
(as well as $H_p(K_a)$)



What are important bits? ("cone off" K^{a_0})
(revisited)

$H_0(K^*)$ minus 1



Extended persistence Module

regular persistence

$$H_p(K_{a_1}) \rightarrow H_p(K_{a_2}) \rightarrow \dots \rightarrow H_p(K_{a_n})$$

$$\hookrightarrow H_p(K, K^{a_n}) \rightarrow \dots \rightarrow H_p(K, K^{a_1})$$

relative
superlevel
set homology

3 kinds of points:

- Ordinary : usual in PD
- Relative : born & die on way down
- Extended : use both

$$H_p(K_{a_1}) \rightarrow H_p(K_{a_2}) \rightarrow$$

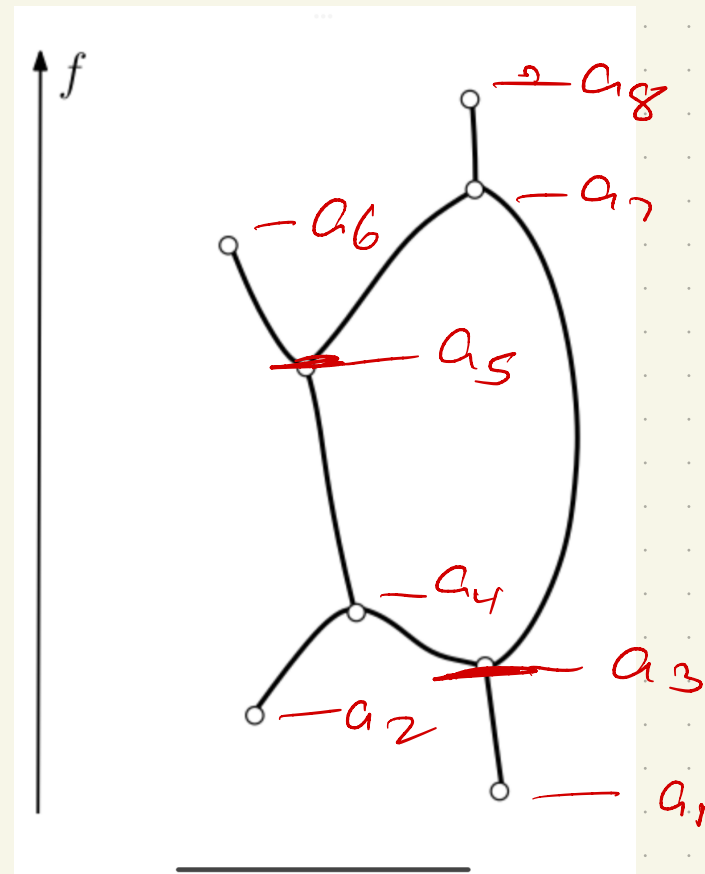
$$\dots \rightarrow H_p(K_{a_8})$$

$$\hookrightarrow H_p(K, K^{a_8}) \rightarrow H_p(K, K^9)$$

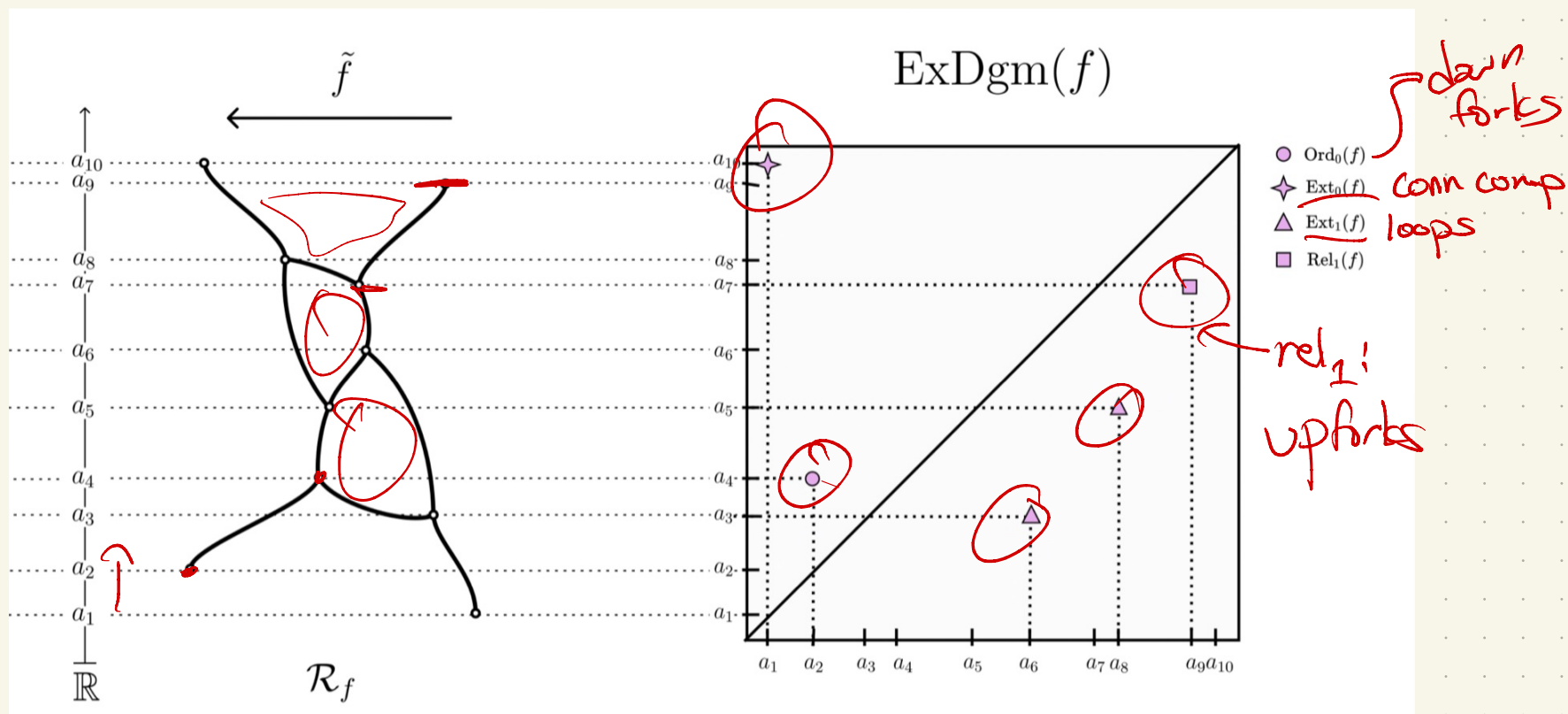
$$\hookrightarrow H_p(K, K^{a_6}) \rightarrow H_p(K, K^{a_5})$$

$$\langle \beta \rangle \longrightarrow 0$$

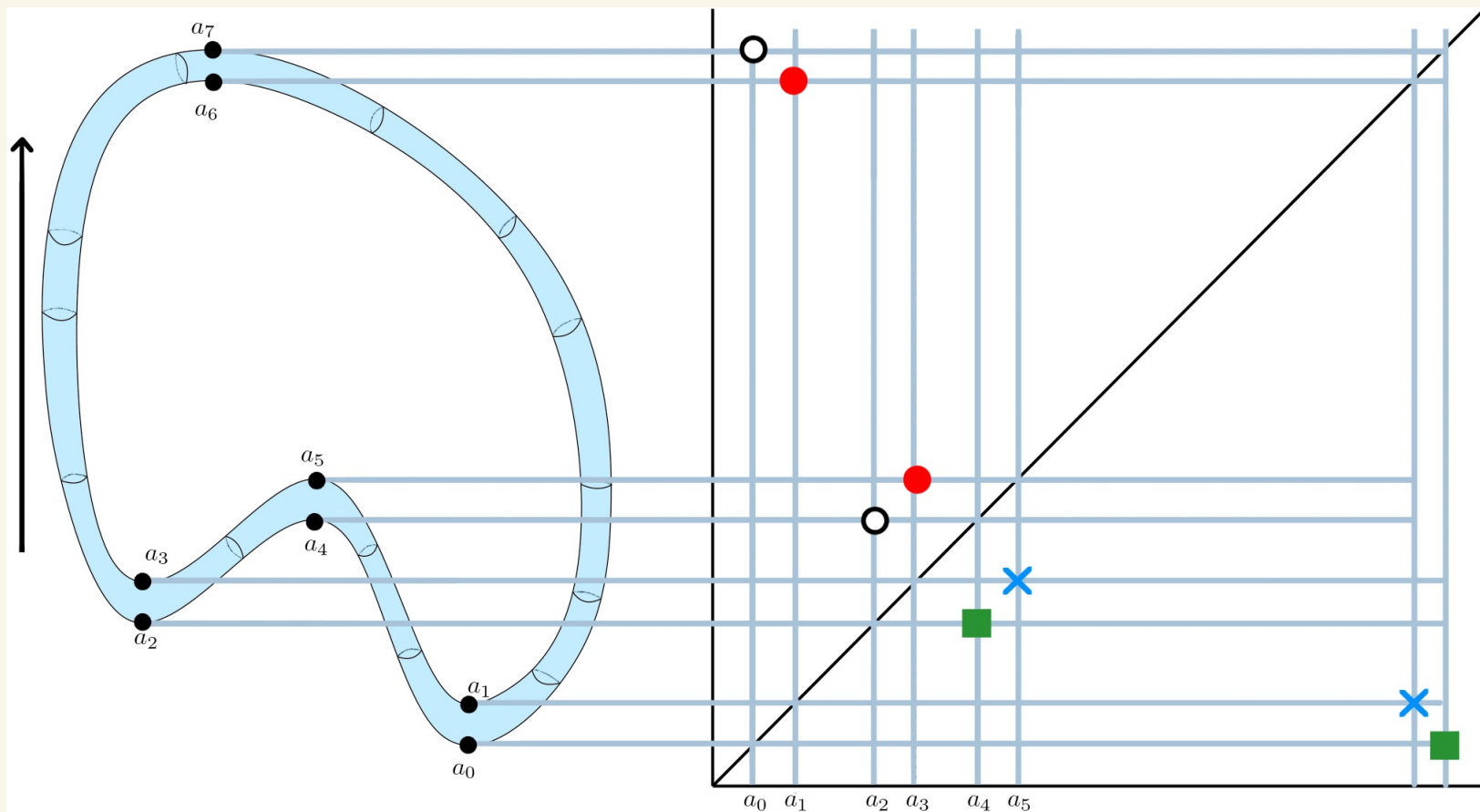
$$\hookrightarrow \dots H_p(K, K^{a_3})$$



On graphs



More generally: on manifolds get symmetry



• H_0
(ordinary & extended)

• ordinary H_1

▣ relative H_2

× relative H_1

Under the hood:

• Very beautiful combination of Lefschetz & Poincaré duality (that 2010 paper)

• Some more algebraic connections

Turner-Robins-Morgan
2022

Reeb Graphs (named for Georges Reeb, 1946, by René Thom)

Recall: level sets

Given (PL) $f: |K| \rightarrow \mathbb{R}$,

a level set is $f^{-1}(a)$

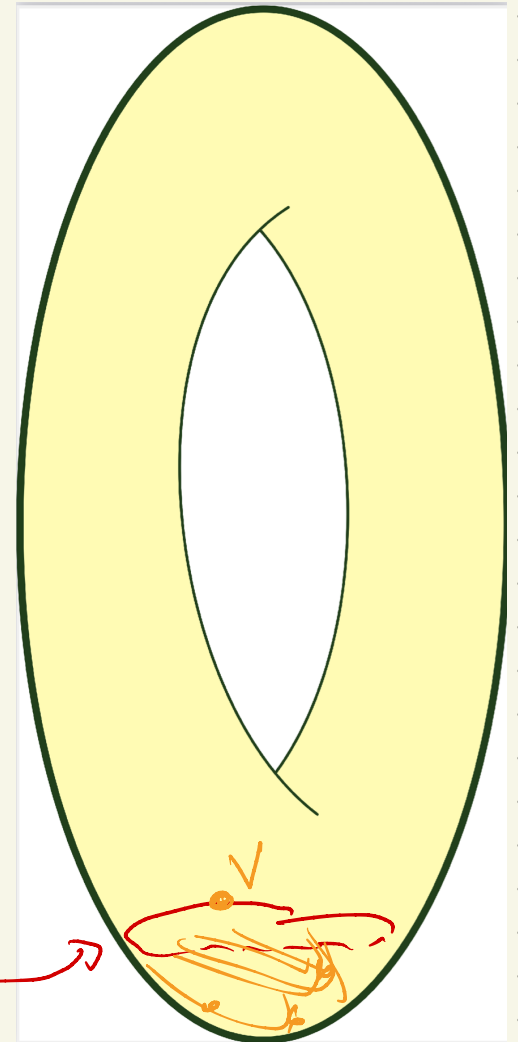
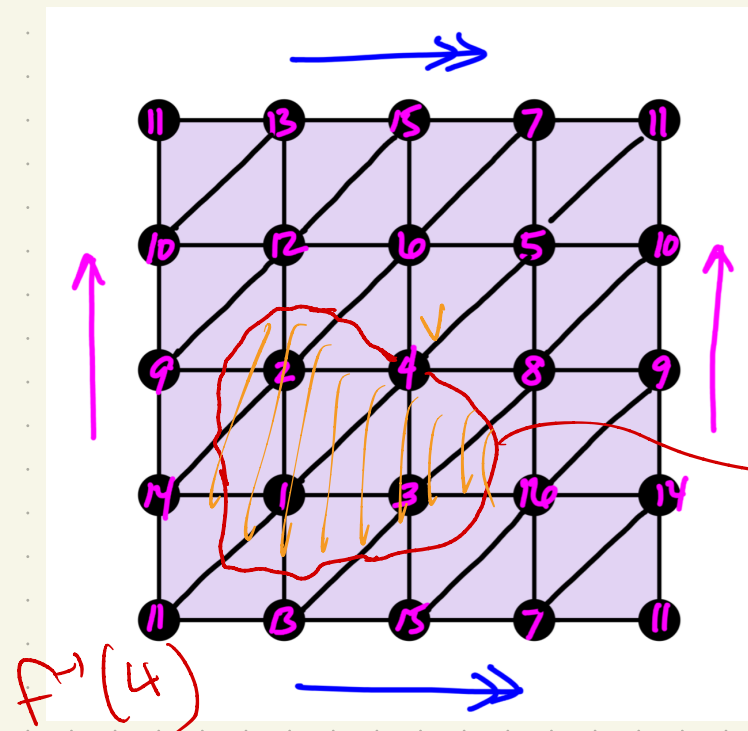
for $a \in \mathbb{R}$

Ex:

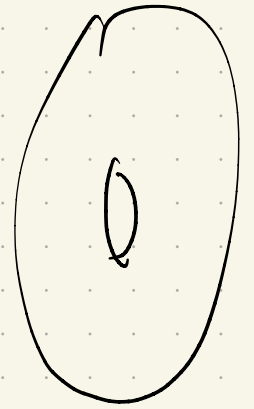
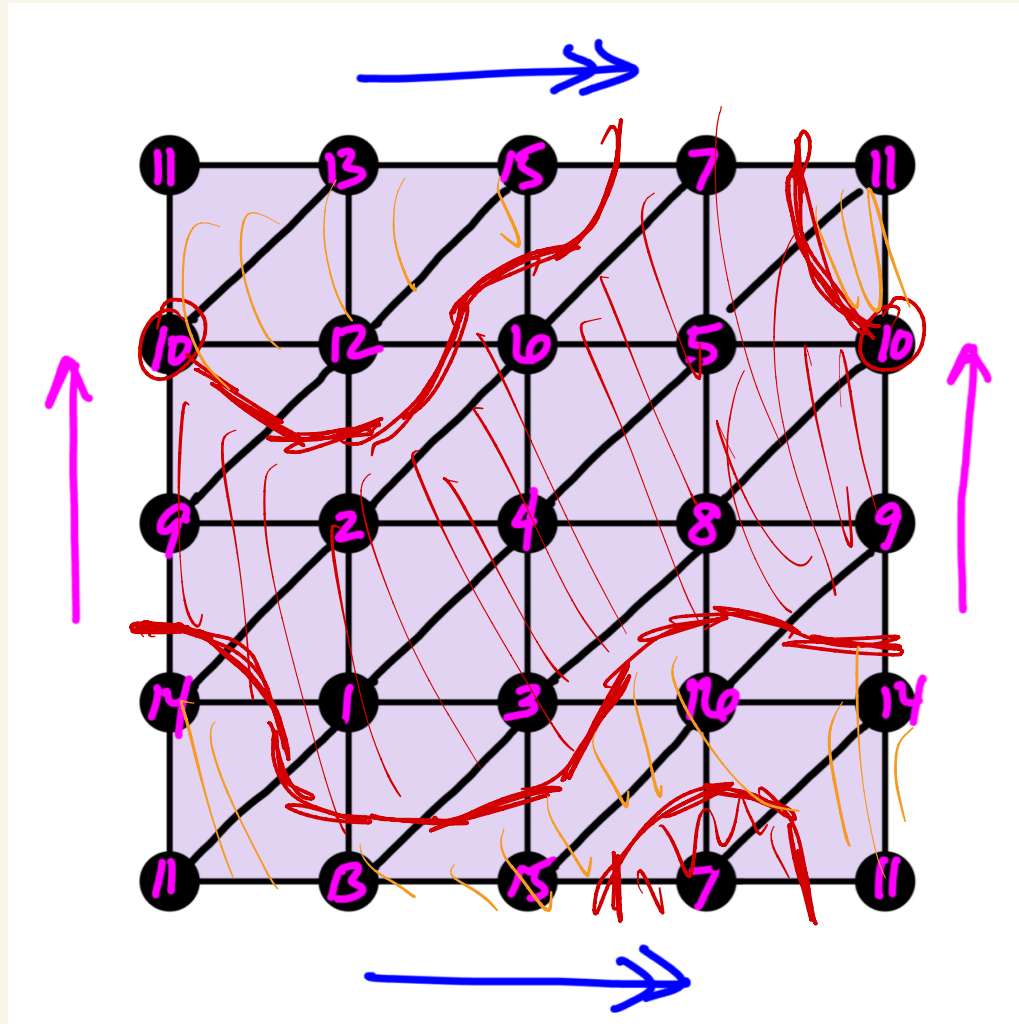
Consider

$$f^{-1}((-\infty, 4])$$

$$\& f^{-1}(4) \leadsto$$



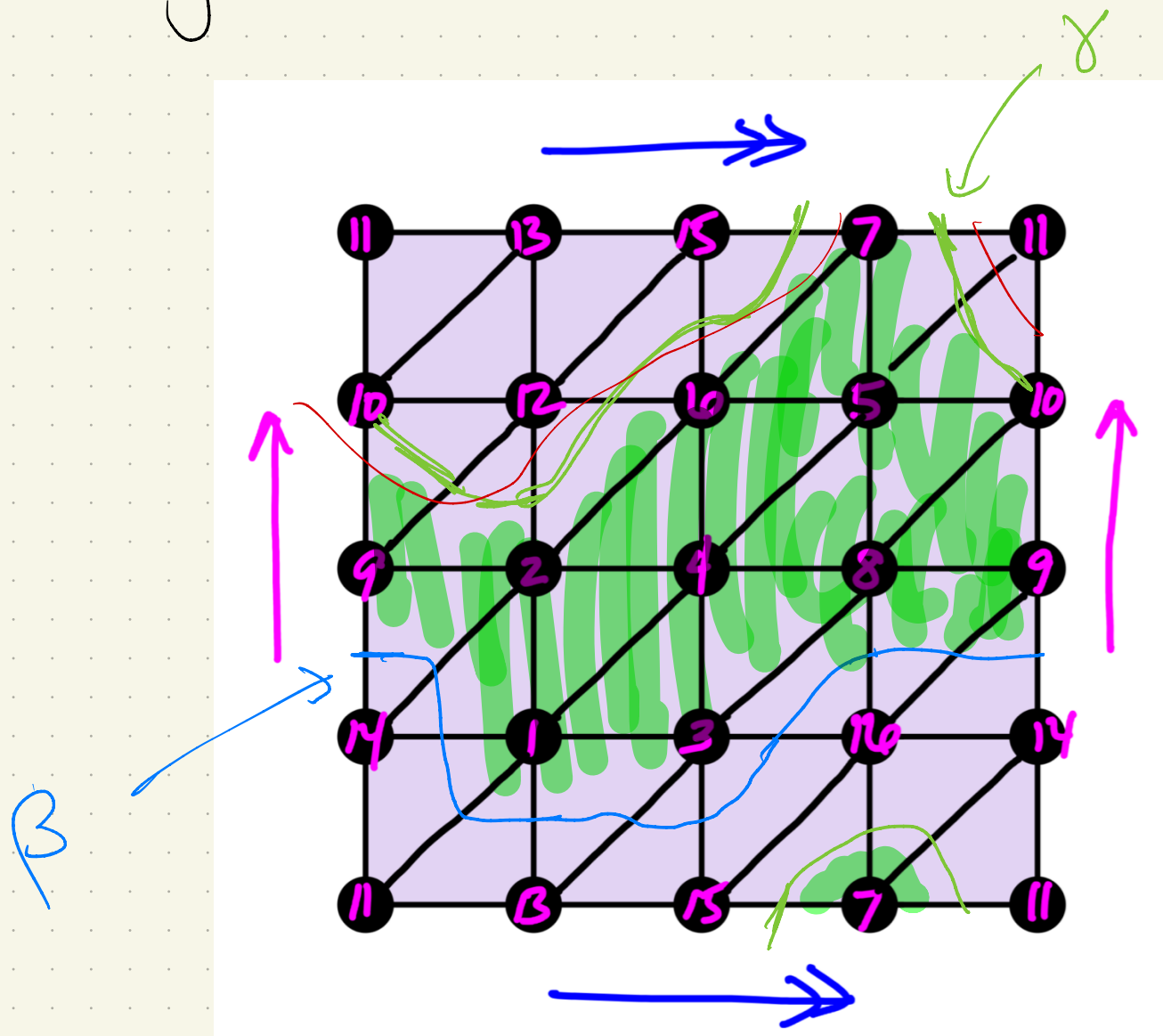
Practice: What is $f^{-1}(10)$?



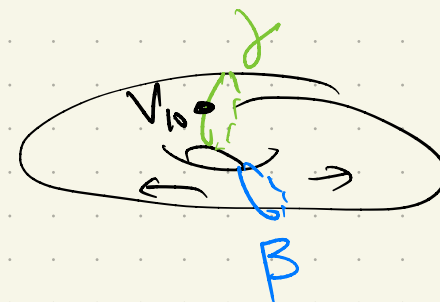
$$f^{-1}((-\infty, 10))$$

$$f^{-1}([10, \infty))$$

Answer: just in case! :)



On
Torus!



Sublevel set $f^{-1}(\underline{-\infty}, 10]$

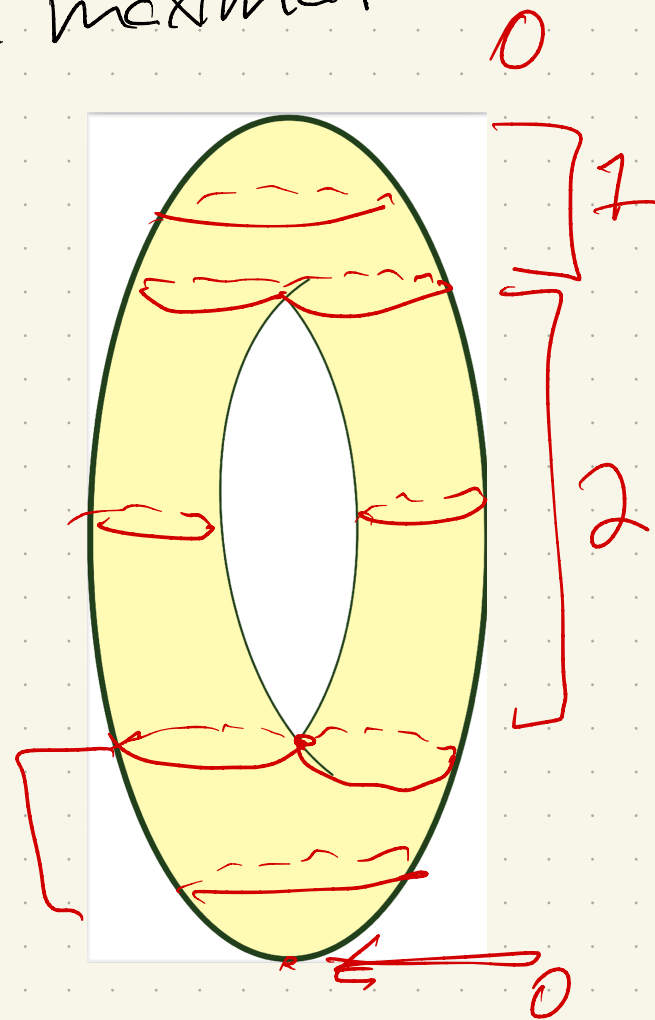
Recall:

A space is disconnected $\Leftrightarrow \exists$ disjoint open sets U, V with $X = U \cup V$
 $\hookrightarrow$ otherwise, connected.

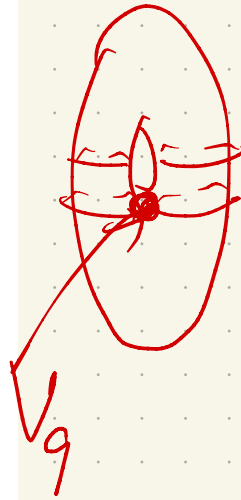
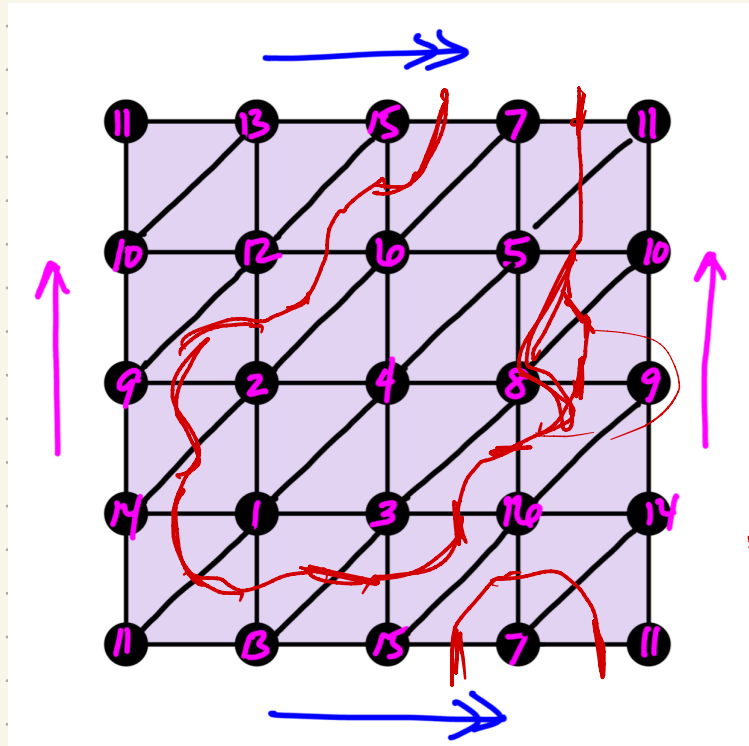
A **connected component** is a maximal subset that is connected

A **contour** is a connected component of a level set

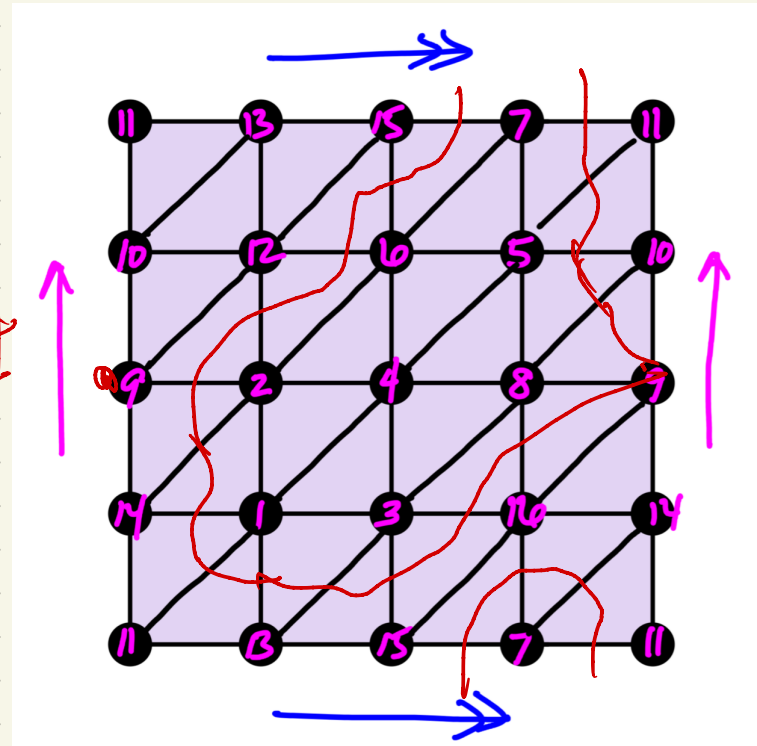
Q: How do contours change a level set changes?



Back to triangulation:
 $f^{-1}(8)$



$f^{-1}(9)$



What shape, & how many components?

(& recall $f^{-1}(10)$ a few slides back had 2 contours)

Equivalence relations & quotients

A binary relation $\sim$ on a set T is an equivalence relation if $\forall a, b, c \in T$

- $a \sim a$ [reflexive]
- $a \sim b \Leftrightarrow b \sim a$ [symmetric]
- if $a \sim b$ & $b \sim c \Rightarrow a \sim c$ [transitive]

Equivalence class of a under $\sim$

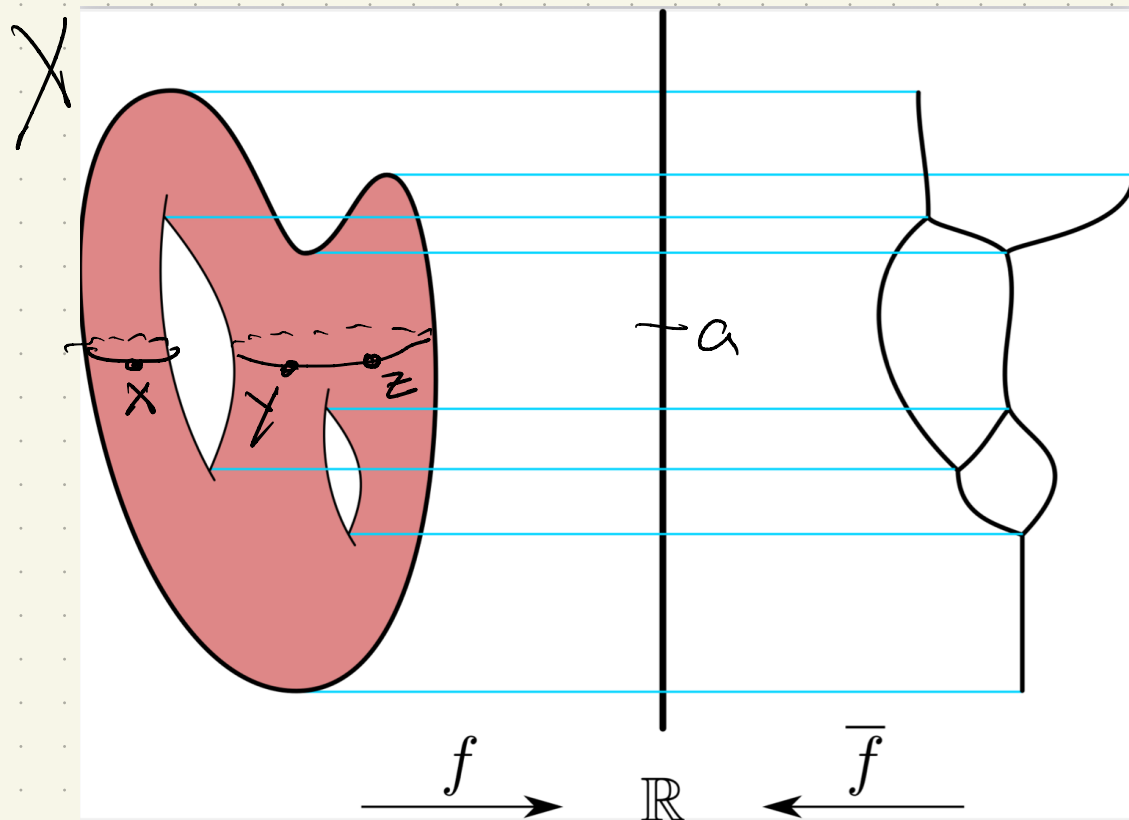
$$[a] = \{x \in T \mid x \sim a\}$$

On a topological space with an equivalence relation, get quotient map $q(a) = [a]$

(and U is open $\Leftrightarrow q^{-1}(U)$ is open)

Reeb graph equivalence relation

Given function $f: X \rightarrow \mathbb{R}$.

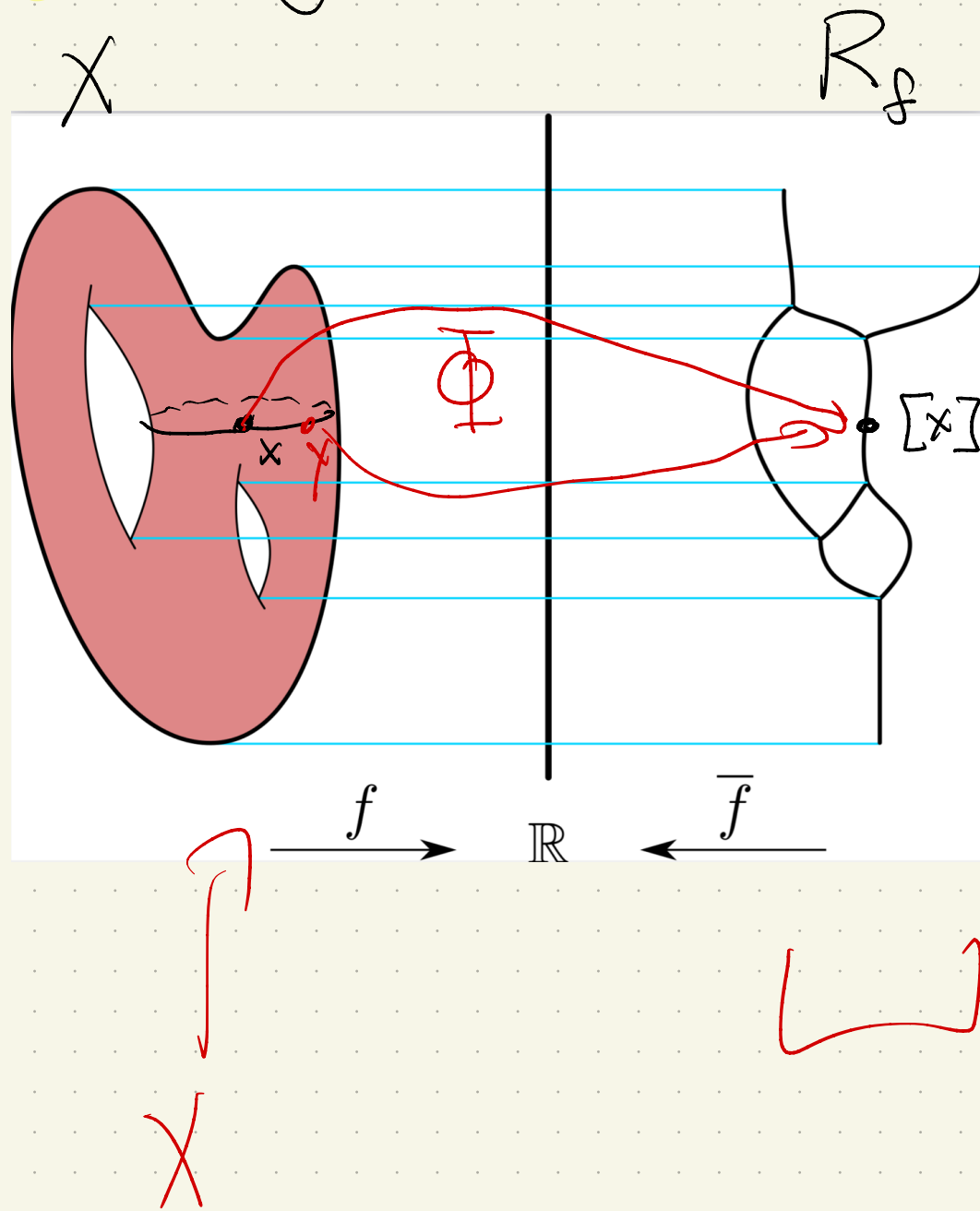


Define equiv. rel. $\sim$
by $x \sim y$ iff:

- $f(x) = f(y) = a$
- x & y are in same connected component of level set $f^{-1}(a)$.

$$\begin{aligned} x &\sim y, z \\ y &\sim z \end{aligned}$$

Reeb graphs



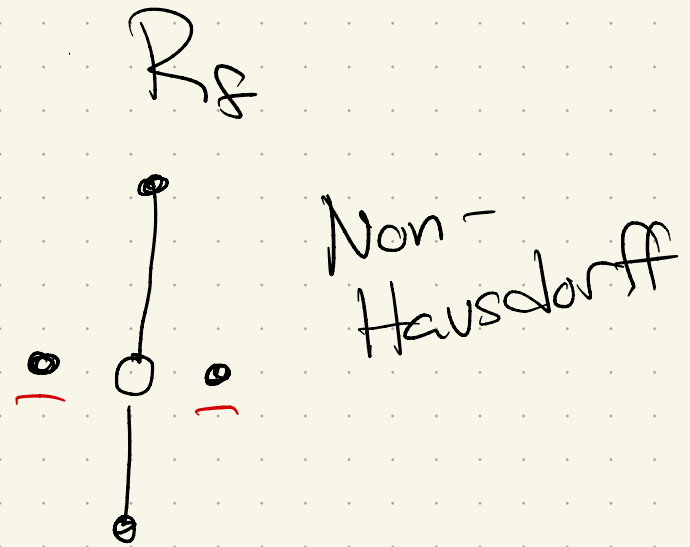
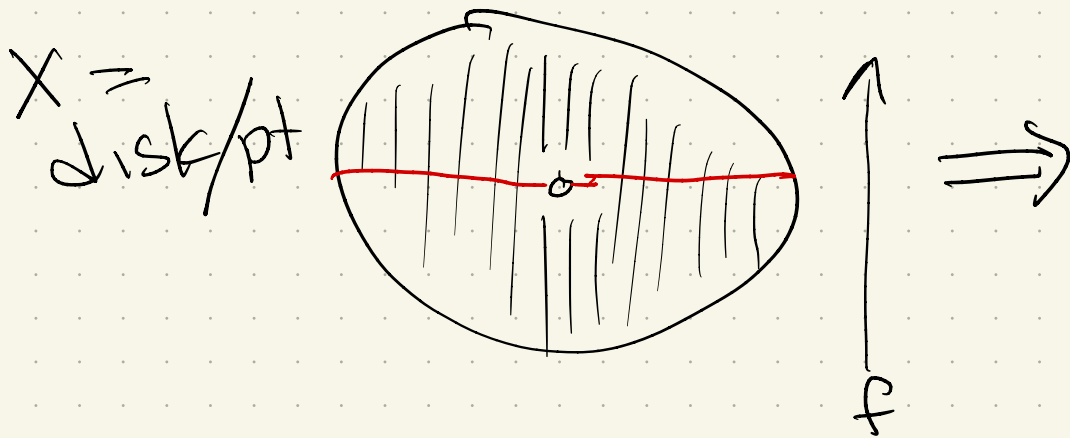
- Let $[x]$ denote the equivalence class of $x \in X$.
- The Reeb graph $R_f = R(X, f)$ is the quotient space $X / \sim$
- Let $\Phi: X \rightarrow R_f$, $x \mapsto [x]$ be the quotient map

Niceness assumptions

If input is "nice" (both X & f), then R_f is a graph.

- Morse function on compact manifold
- PL function of compact polyhedron
- Others...

Bad example:

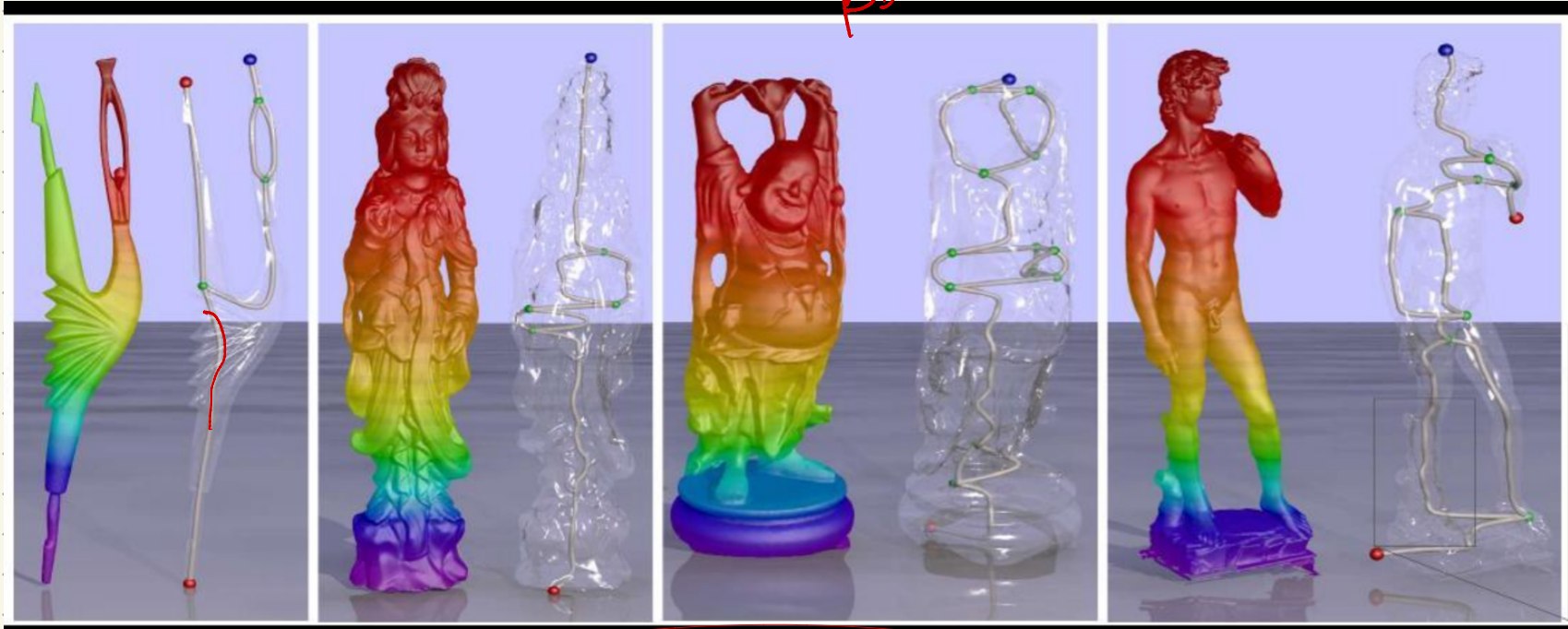


Applications: a topological signature

Pascucci et al

$B_1 = 5$

no $\mathbb{Z}^d$



←
|

Note: a bit of a lie in this figure:

no embedding!
 $\mathbb{R}$ -values

$\mathbb{Z}_f$