TDA - fall 2025

Rank invariants & generalized Betti #s & PDs.

Last time: Representation theory

Directed graphs, Q_0 + Q_1 ← quiver

vertices ↑ edges

↳ Realization:

each vertex gets vector space over field k (eg $\mathbb{Z}_2$)

edge a

$\Downarrow$

$h(a) = v_i$

$t(a) = v_j$

$$
\begin{tikzcd}
\boldsymbol{k} \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[d, "\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[d, "0"'] & \boldsymbol{k} \arrow[l, "\begin{pmatrix}1\\1\end{pmatrix}"'] \arrow[d, "-\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[l, "\begin{pmatrix}0 & 1\end{pmatrix}"'] \arrow[r, "\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0 & -1\end{pmatrix}"'] & \boldsymbol{k}^2 \arrow[d, "0"'] \\
\boldsymbol{k} \arrow[r, "0"] & 0 & \boldsymbol{k} \arrow[l, "0"'] & \boldsymbol{k} \arrow[l, "\mathbb{1}"'] \arrow[r, "0"] & 0
\end{tikzcd}
$$

each edge: linear map

Quiver representations are like vector spaces:

- They contain a 0 object
  ↳ all spaces & maps $= 0$
- They have a bi-product, the direct sum:
  $\mathbb{V} \oplus \mathbb{W}$: spaces $V_i \oplus W_i$, $i \in Q_0$
  maps $V_a \oplus W_a = \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix}$
- Every morphism $\phi : \mathbb{V} \to \mathbb{W}$ has a kernel: $(\ker \phi)_i = \ker \phi_i$
  (as well as image & cokernel)

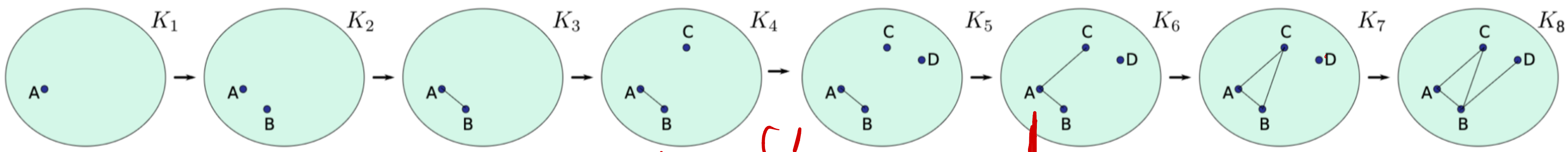
A non-trivial representation $\mathbb{V}$ is called decomposable if it is isomorphic to the direct sum of 2 non-trivial representations.

(Otherwise indecomposable.)

Back to persistence for a minute...

$$\mathbb{V} = \underbrace{\mathbb{W} \oplus \mathbb{X}}_{\text{neither are } 0}$$

Let $k = \mathbb{Z}_2$ & do H_0:



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$$\mathbb{Z}_2 \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{Z}_2^2 \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} \mathbb{Z}_2 \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{Z}_2^2 \xrightarrow{\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}} \mathbb{Z}_2^3 \xrightarrow{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\end{bmatrix}} \mathbb{Z}_2^2 \xrightarrow{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}} \mathbb{Z}_2^2 \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} \mathbb{Z}_2$$

$\boxed{\text{decompose}}$

$$\begin{array}{ccccccccccccccc}
 & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{1} \mathbb{Z}_2 \\
\oplus & 0 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{0} & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to 0 \\
\oplus & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} 0 \\
\oplus & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{0} 0
\end{array}$$

Central question

Classify representations of a given quiver up to isomorphism.

[usually all finite + finite dim]

Define $\underline{\dim} \mathbb{V} = (\dim V_1, \dots \dim V_n)^T$ ← a vector

and $\dim \mathbb{V} = \sum_{i=1}^{n} \dim V_i$

on computers!

How to get a handle on this?

even when dim is finite, these get "wild"

Krull-Remak-Schmidt Theorem

Wedderburn 1909, Remak 1911
Schmidt 1913, Krull 1925

Assuming Q is finite, for any $\mathbb{V} \in rep_k(Q)$, $\exists$ indecomposable representations $\mathbb{V}_1, \dots \mathbb{V}_r$ s.t. $\mathbb{V} = \mathbb{V}_1 \oplus \dots \oplus \mathbb{V}_r$.

just saw w/ barcodes

Moreover, for any other indecomposable rep. $\mathbb{W}_1, \dots, \mathbb{W}_s$ with $\mathbb{V} = \mathbb{W}_1 \oplus \dots \oplus \mathbb{W}_s$ must have $r = s$ and the $\mathbb{W}_i$'s + $\mathbb{V}_j$'s are permutations.

So → to classify, need to understand & characterize indecomposables.

Gabriel's theorem 1972

Let Q be finite quiver & k a field. Then, Q has a finite # of classes of indecomposables $\Longleftrightarrow$ Q is Dynkin.

Why surprising?

k & $\Leftrightarrow$ are irrelevant

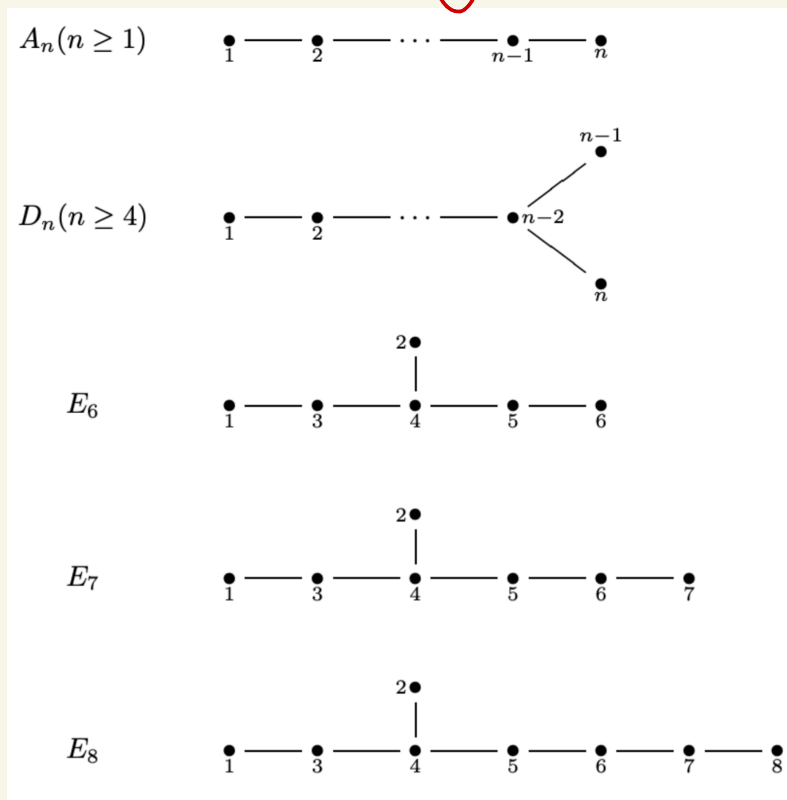
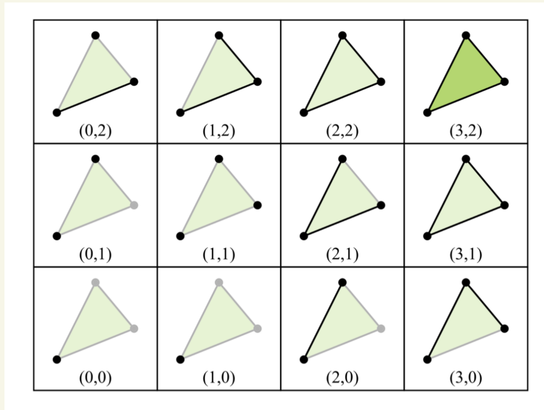


FIGURE A.2. The Dynkin diagrams.

Back to persistence

Why we still care?



$\Longrightarrow$

not Dynkin

$$\begin{tikzcd}
k^2 \arrow[r] & k \arrow[r] & k \arrow[r] & k \\
k^2 \arrow[r] \arrow[u] & k^3 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u] \\
k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}$$

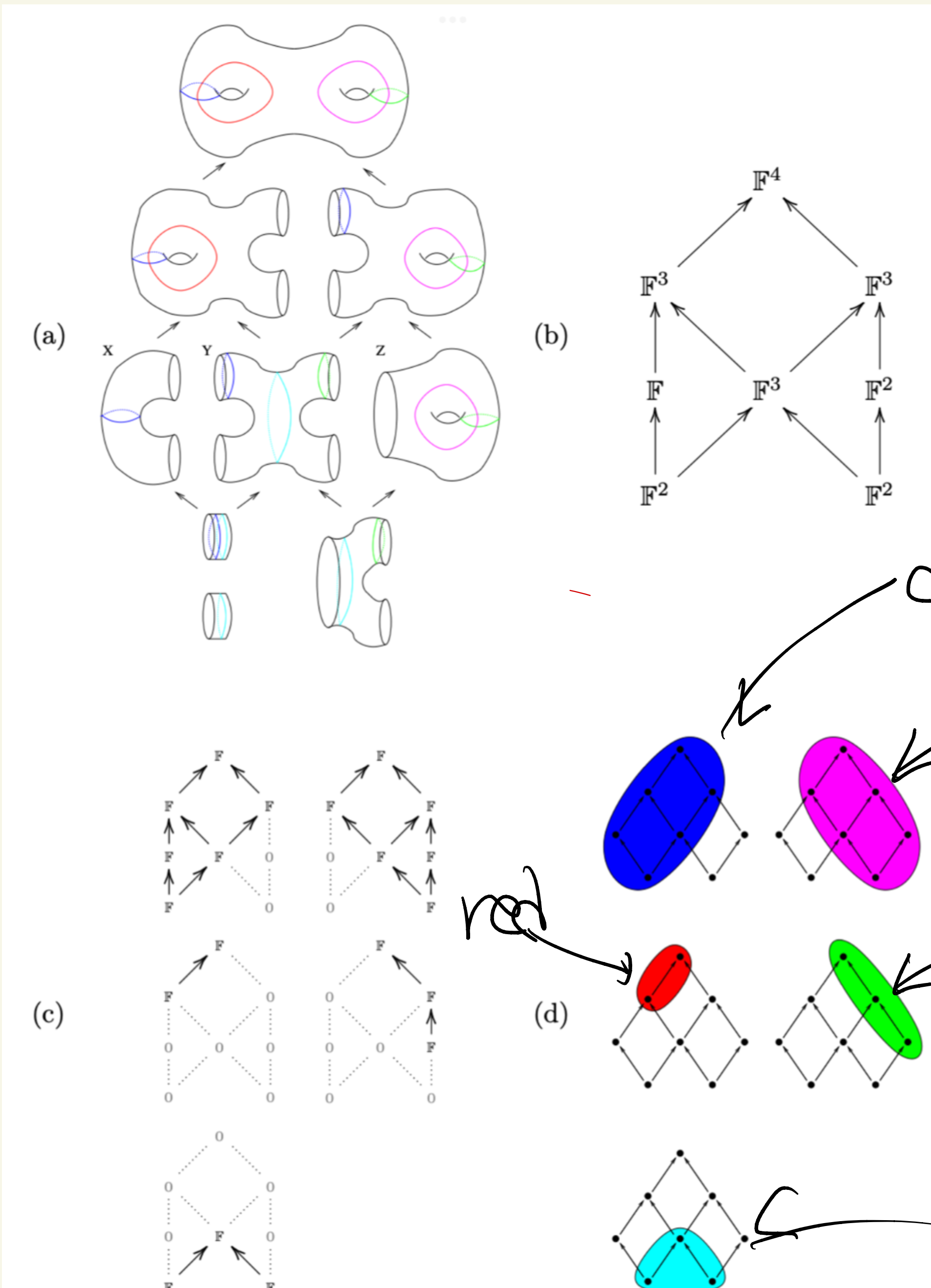
since <u>not</u> Dynkin in general, Gabriel's theorem doesn't apply $\Rightarrow$ Q has infinite # of isomorphism classes of indecomposables.

Result: no barcodes

Invariants

Carrier subgraph:

C.-Letscher 2019



Up to choice on generators:

dark blue cycle

green

pink

red

turquoise

Biparameter filtrations again

① Dimension Function

Simply map each $a \in \cancel{\mathbb{R}}\,\mathbb{Z}^2$ to $\dim(M_a)$:

$$
\begin{tikzcd}
k^2 \arrow[r] & k \arrow[r] & k \arrow[r] & k \\
k^2 \arrow[r] \arrow[u] & k^3 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u] \\
k \arrow[r, "\text{id?}"'] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}
\implies
\begin{matrix}
2 & 1 & 1 & 1 \\
2 & 3 & 1 & 1 \\
1 & 1 & 1 & 1
\end{matrix}
$$

pros: easy to compute + visualize!

cons: no notion of lifespan or persistence

② Rank invariant

For each linear map $M_a \to M_b$, compute $\operatorname{rank}(M_a \to M_b)$.

Note: lots of these!

$$
\begin{tikzcd}
\boldsymbol{k} \arrow[r, "\binom{1}{0}"] \arrow[d, "\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[d, "0"'] & \boldsymbol{k} \arrow[l, "\binom{1}{1}"'] \arrow[d, "-\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[l, "(0\ 1)"'] \arrow[r, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"] \arrow[d, "(0\ -1)"] & \boldsymbol{k}^2 \arrow[d, "0"'] \\
\boldsymbol{k} \arrow[r, "0"] & 0 & \boldsymbol{k} \arrow[l, "0"'] & \boldsymbol{k} \arrow[l, "\mathbb{1}"'] \arrow[r, "0"] & 0
\end{tikzcd}
$$

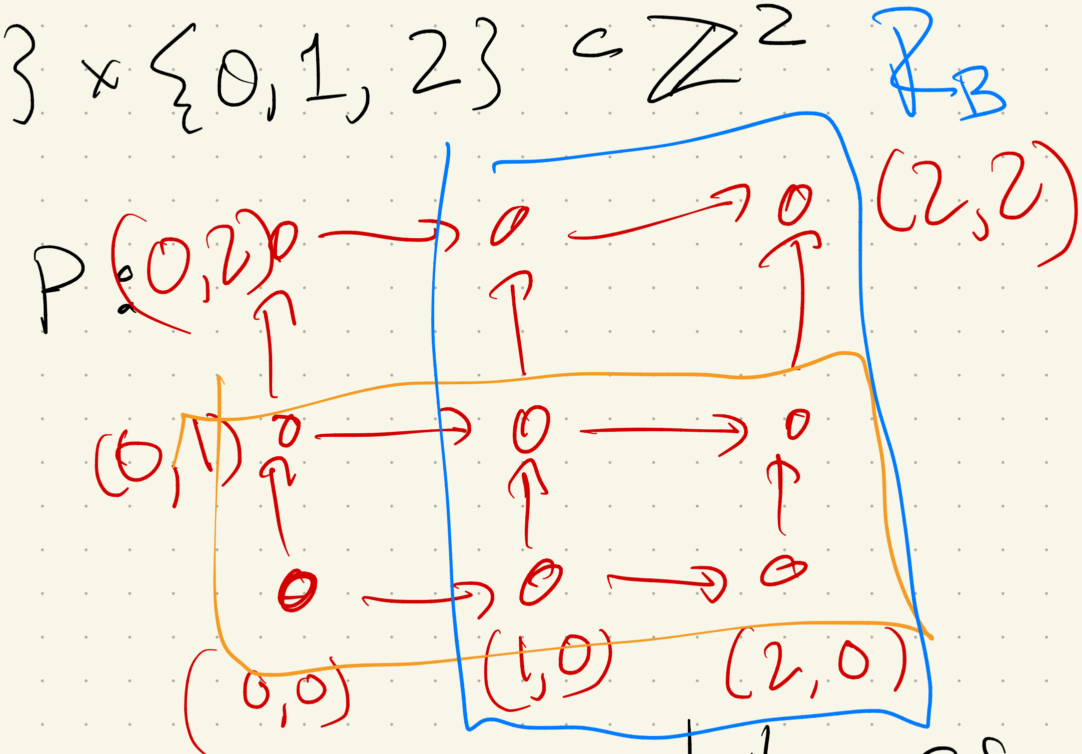
bipersistence modules $n \times n$ grid $\to$ n^3 in size

An example to illustrate issues + concept:

Let $P = \{0,1,2\} \times \{0,1,2\} \subset \mathbb{Z}^2$

with usual partial order:

$(i,j) \le (i',j') \iff i \le i' \;\&\; j \le j'$



Let's build a bipersistent module as direct sum of 2 rectangles:

$R_A = \{(i,j) \mid i \in \{0,1,2\},\ j \in \{0,1\}\}$

$R_B = \{(i,j) \mid i \in \{1,2\},\ j \in \{0,1,2\}\}$

Example continued:

For each $p = (i, j)$,

$$\underline{A(p)} = \begin{cases} K & \text{if } p \in R_A \\ 0 & \text{if not} \end{cases}$$

$$\underline{B(p)} = \begin{cases} K & \text{if } p \in R_B \\ 0 & \text{if not} \end{cases}$$

& all maps either $\underline{0}$ or $\underline{1}$

Let $M = \underline{A \oplus B}$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \\
K \arrow[u, "0"] \arrow[r, "1"] & K \arrow[u, "0"] \arrow[r, "1"] & K \arrow[u, "0"] \\
K \arrow[u, "1"] \arrow[r, "0"] & K \arrow[r, "1"] & K
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r, "0"] & K \arrow[r] & K \\
0 \arrow[u, "0"] \arrow[r, "0"] & K \arrow[u] \arrow[r, "1"] & K \arrow[u] \\
0 \arrow[u, "0"] \arrow[r, "0"] & K \arrow[u] \arrow[r, "1"] & K
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & K \\
K \arrow[u] \arrow[r] & K^2 \arrow[u] \arrow[r] & K^2 \arrow[u] \\
K \arrow[u] \arrow[r] & \underset{\cong K \times K}{K^2} \arrow[u] \arrow[r] & K^2 \arrow[u]
\end{tikzcd}$$

dimension grid here:

$$\begin{array}{ccc} 0 & k & k \\ k & k^2 & k^2 \\ k & k^2 & k^2 \end{array} \Longrightarrow \begin{array}{ccc} 0 & 1 & \boxed{1} \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{array}$$

Rank invariant: $\forall\, p \leq q$, defined as

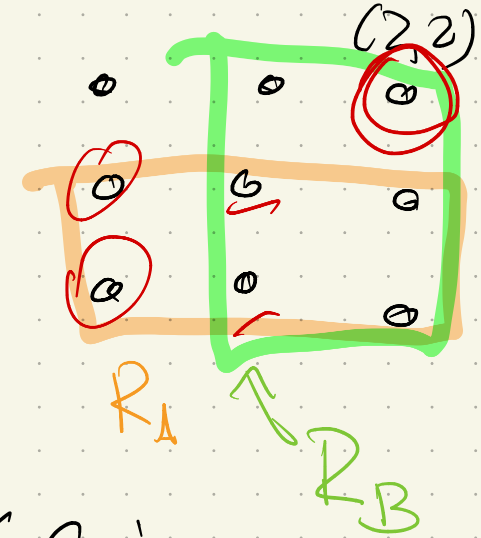
$$\mathrm{rank}_M(p,q) = \mathrm{rank}(p \to q)$$

Here, $\mathrm{rank}(p \to q) =$ dim of map

So, trying to compute:

fix $q = (2,2)$

$q \notin R_A$, but $q \in R_B$



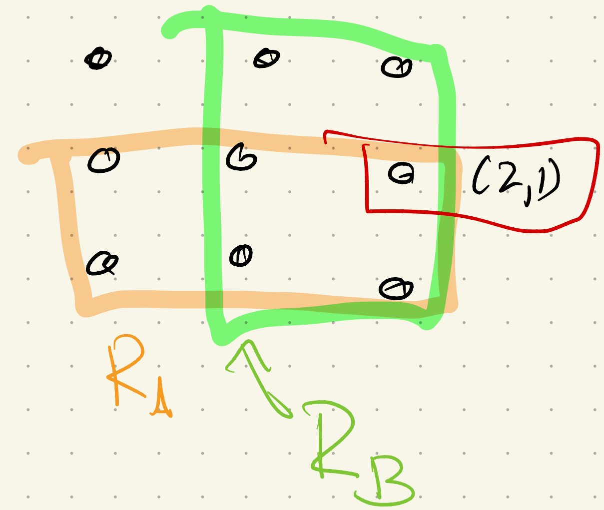
Then consider all p s.t. $p \leq q$:

$$\mathrm{rank}(p \to q) = \underbrace{\mathrm{rank}_A(p \to q)}_{\text{here:}} + \mathrm{rank}_B(p \to q)$$

so: $\mathrm{rank}_q(p) =$

0	1	
1	2	2
1	2	2

Another: fix $q = (2,1)$



Now, all $p \le q$!

$\mathrm{rank}_q(p) =$

1	2	
1	2	2

0

$= \mathrm{rank}_A(p \to q)$

$+ \mathrm{rank}_B(p \to q)$

This still (in a sense) measures
"homological features in $\underline{p}$ that
persist until $\underline{q}$"

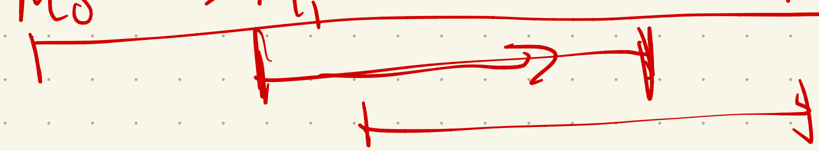
But: non-isomorphic modules can
share rank invariants
& can't have "good barcodes"

Let's unpack why...

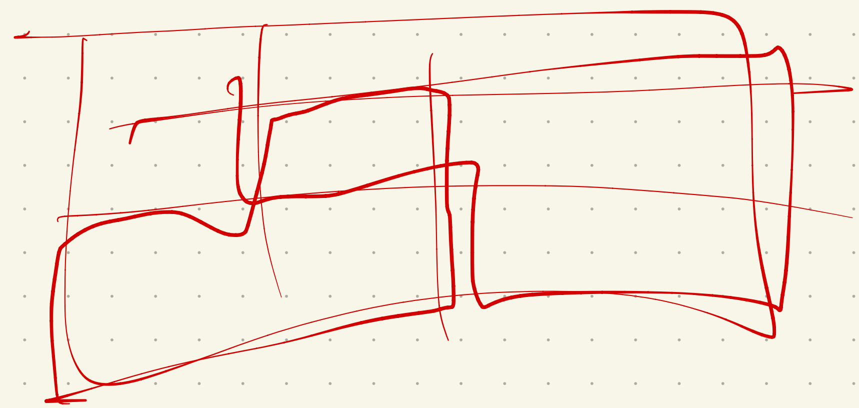
"Good" barcodes: what we mean

Consider a multiset of subsets of $P \rightsquigarrow$ a "barcode"

in standard: $M_0 \to M_1 \to \cdots \to M_n$



in multi-d:



Say it is good if $\forall\, x \leq y$,

$$\mathrm{Rank}(M_x \to M_y) = \left|\{ S \in B \mid x, y \in S \}\right|$$

↳ # elements w/ x & y

2d modules are <u>not</u> good:

$\boxed{\text{Botnan-Lesnick 2023}}$

$$\begin{tikzcd}
k \arrow[r, "="] & k \arrow[r, "0"] & 0 \\
k \arrow[u, "="] \arrow[r, "{[1,0]^T}"] & k^2 \arrow[u, "{[1,0]}"'] \arrow[r, "{[1,1]}"] & k \arrow[u, "0"'] \\
0 \arrow[u, "0"] \arrow[r, "0"'] & k \arrow[u, "{[0,1]^T}"] \arrow[r, "="] & k \arrow[u, "= \ \mathrm{id}"']
\end{tikzcd}$$

An example: Suppose B is a good barcode:

Know:

$$\mathrm{Rank}(M_{(0,1)} \to M_{(2,1)}) = \mathrm{Rank}(M_{(0,1)} \to M_{(1,2)})$$
$$= \mathrm{Rank}(M_{(1,0)} \to M_{(2,1)}) = 1$$

What must B contain?

at least one set per pair listed

$$
\begin{tikzcd}
k \arrow[r, "="] & k \arrow[r] & 0 \\
k \arrow[u, "="] \arrow[r, "{[1,0]^T}"] & k^2 \arrow[u, "{[1,0]}"] \arrow[r, "{[1,1]}"] & k \arrow[u] \\
0 \arrow[u] \arrow[r] & k \arrow[u, "{[0,1]^T}"] \arrow[r, "="] & k \arrow[u, "="]
\end{tikzcd}
$$

$= k \times k$

Need

$(0,1), (2,1) \in I$

$(0,1), (1,2) \in J$

$(1,0), (2,1) \in K$

& $I, J, K \in B$

But $\dim M_{0,1} = \dim M_{2,1} = 1$

$\Rightarrow$ $I = J = K$ & all # pairs included

Contradiction!

What is $\mathrm{Rank}(M_{(0,1)} \to M_{(1,2)})$? 0

Any barcode B lacks this "goodness"

Note: This is a module whose rank invariant is not equal to the rank invariant of any interval-decomposable module.

But: is the difference between rank invariants of 2 interval decomposable modules!

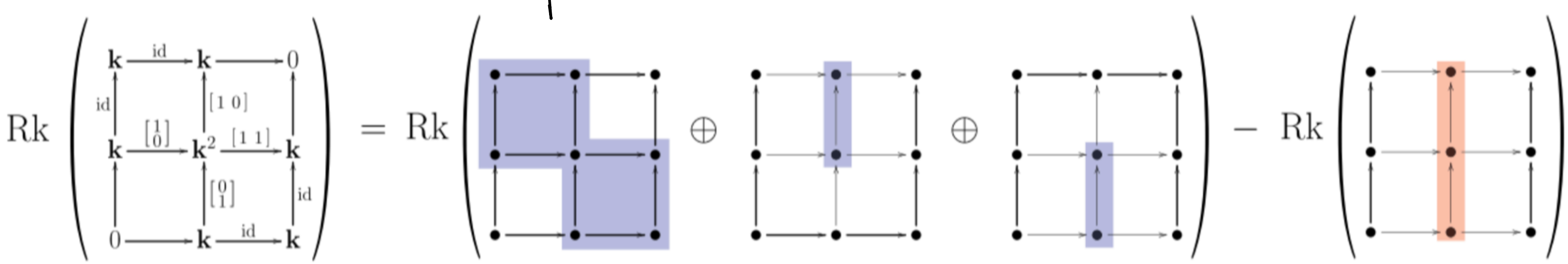
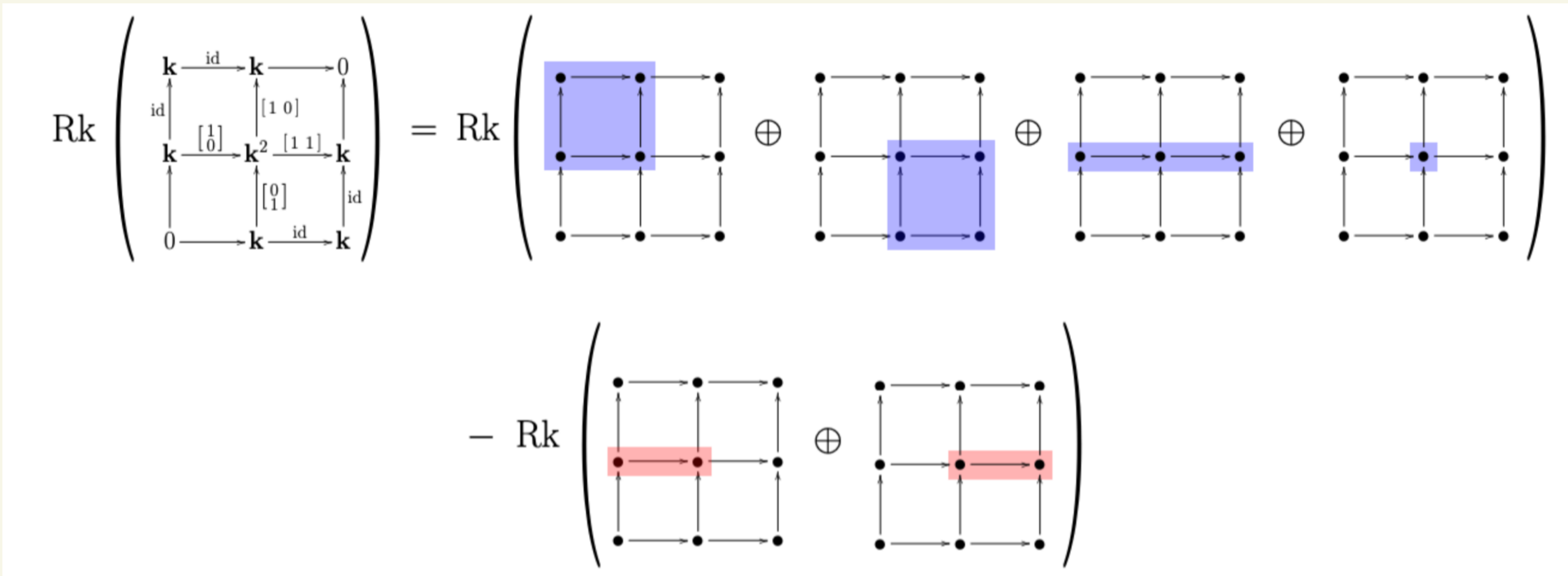


Fig. 2 The indecomposable module M on the left-hand side does not have the same rank invariant as any direct sum of interval modules on the 3×3 grid. However, Rk M is equal to the difference between the rank invariants of two direct sums of interval modules, as shown on the right-hand side. Blue is for intervals counted positively in the decomposition, while red is for intervals counted negatively (Color figure online)

This can be useful

↳ but not unique:



(3) Bigraded Betti Numbers

Essentially, for each $(a,b) \in \mathbb{N}^2$, record new generators or relations that appear at (a,b).

normal persistence

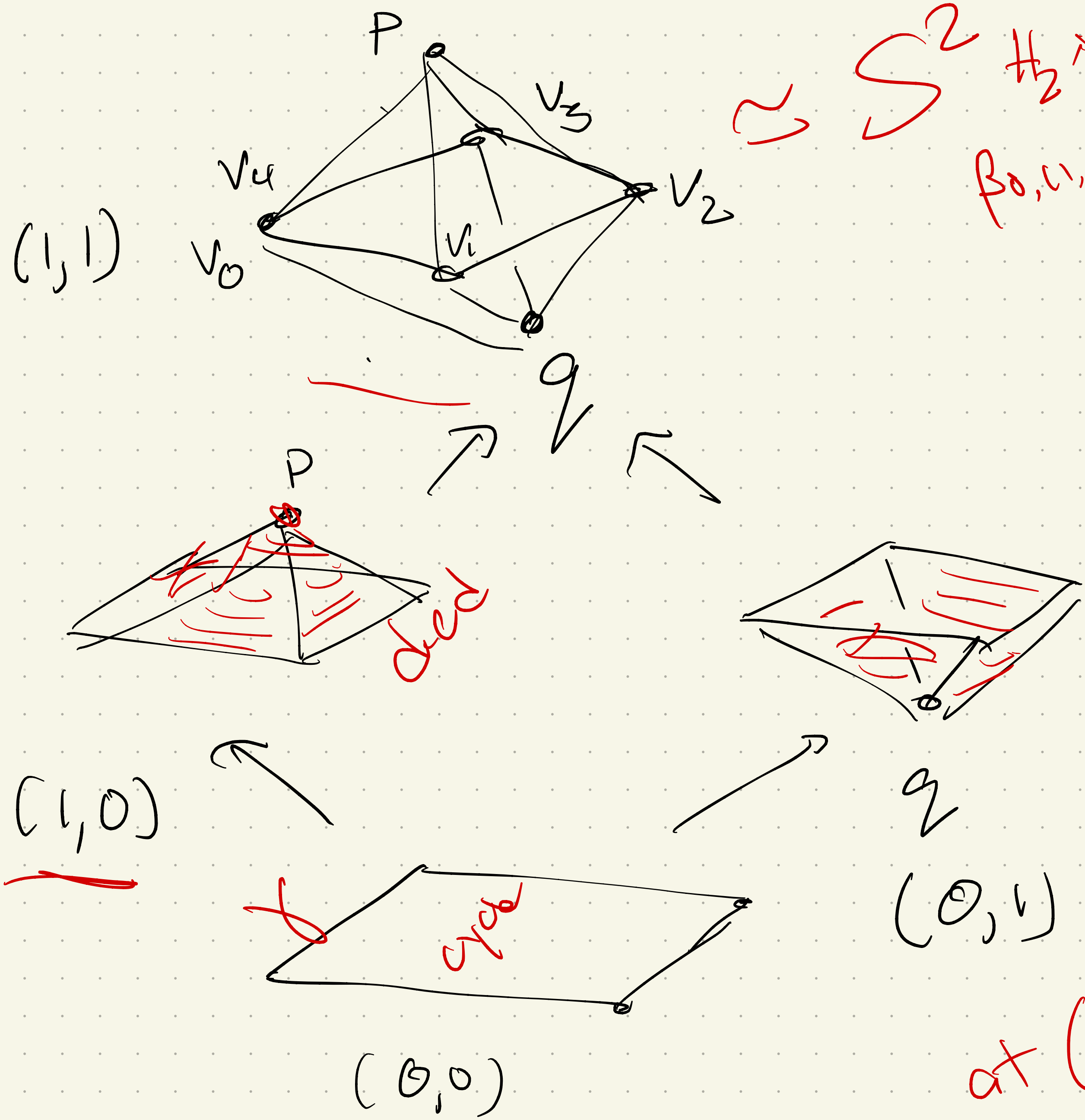
$\beta_{0,(a,b)}$ = # of new homology classes born

$\beta_{1,(a,b)}$ = # of independent relations (ie deaths) at (a,b)

new → $\beta_{2,(a,b)}$ = # of relations among relations at (a,b)

etc.

What is β_2? More complex relations!



$H_1 + \beta_i$:

β_0	β_1	β_2

at (1,1): $\beta_{2,(1,1)} = 1$
b/c 2 death are same

at (0,1): death
$\beta_{1,(0,1)} = 1$

at (1,0): death
$\beta_{1,(1,0)} = 1$

at (0,0): birth in H_1 $\beta_0 = 1$

New trends connect this to something called the Möbius inversion.

Patel 2018
Patel & Skraba '23

- Uses invariants from classical combinatorics

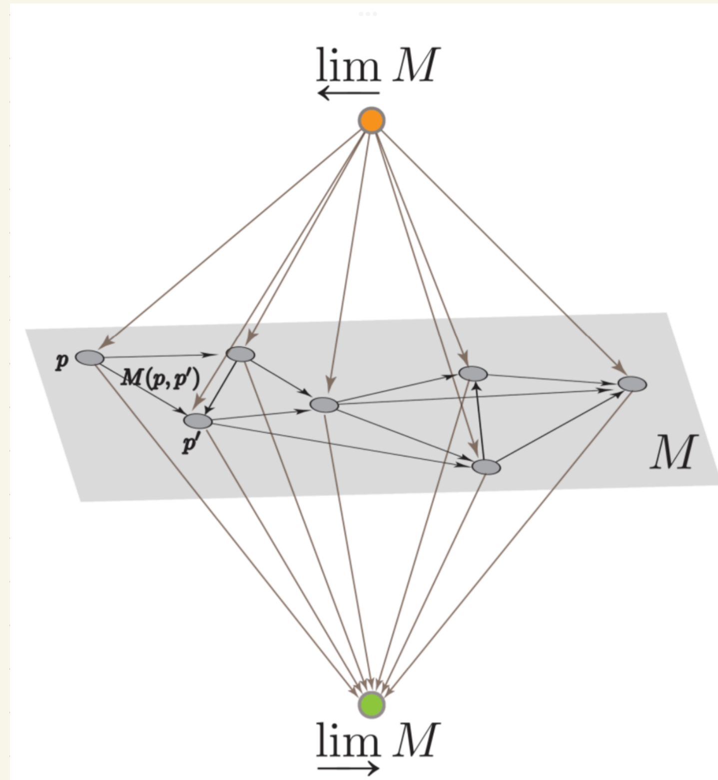


Figure 4. The notions of limit and colimit of a diagram of vector spaces are used in a fundamental way to generalize the notion of rank invariant and persistence diagram.

In addition, algebraic generalizations are being developed

Kim & Memoli 2023

Vizualization: Rivet

Lesnick-Wright 2015

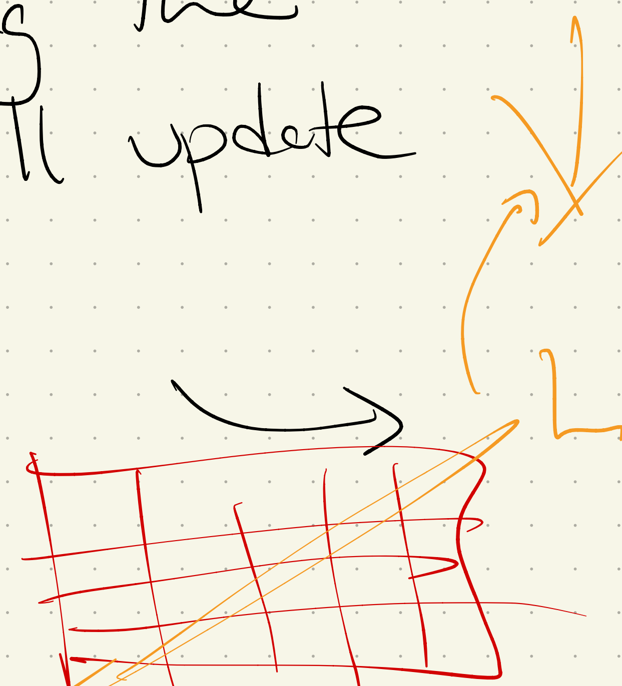
Interactive tool for 2d filtrations.

Bigraded Betti numbers:

- user selects a line $L \in \mathbb{R}^2$
- software displays the barcode $B(M^L)$
- user can then drag the line, + barcode will update

Rips complex land

↳ pick common filtrations



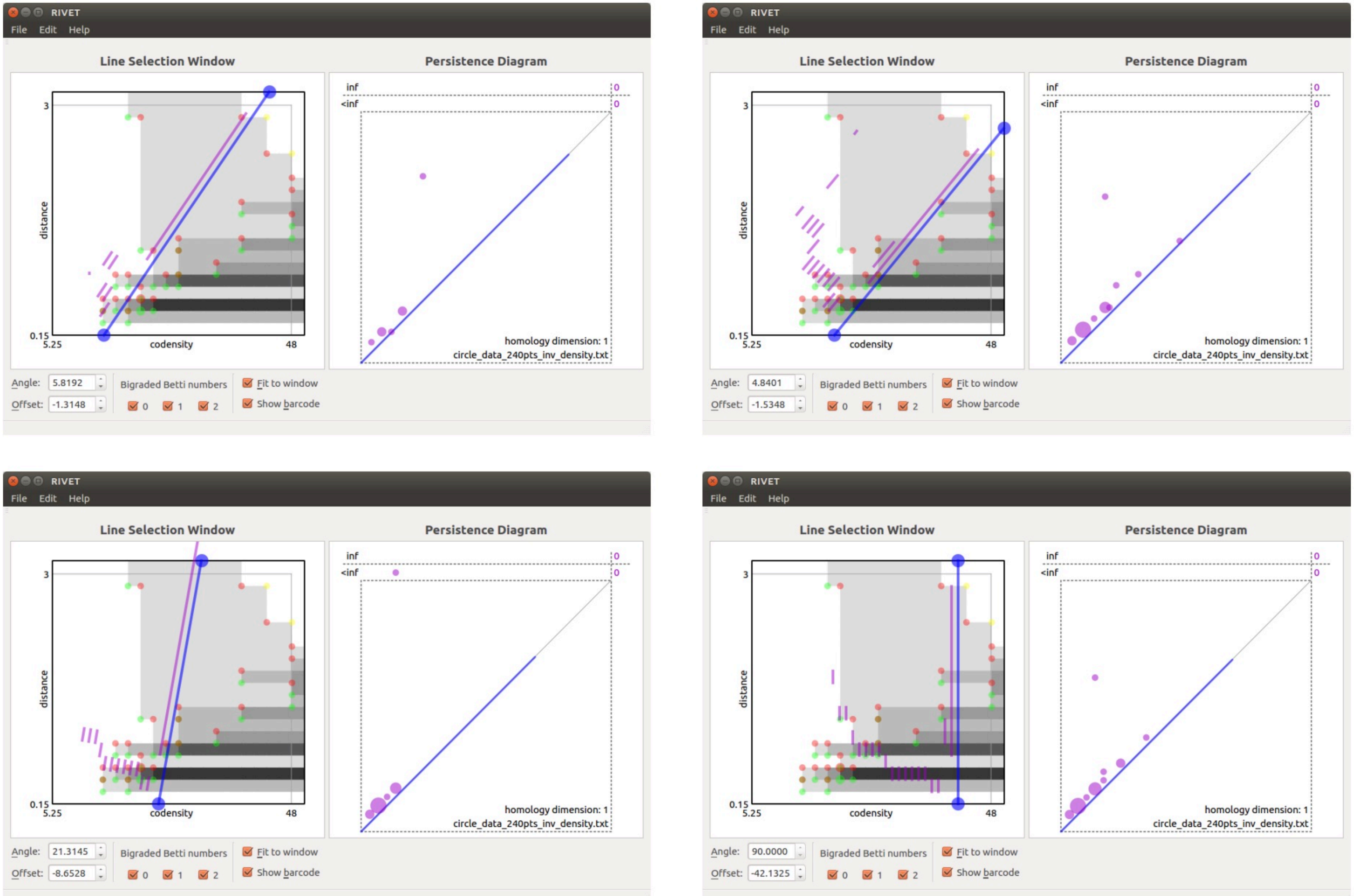


Figure 3: Screenshots of RIVET for a single choice of 2-D persistence module M and four different lines L. RIVET provides visualizations of the dimension of each vector space in M (greyscale shading); the 0^{th}, 1^{st}, and 2^{nd} bi-graded Betti numbers of M (green, red, and yellow dots); and the barcodes of the 1-D slices M^L, for each L (in purple).