TDA - fall 2025

Bipersistence + invariants

Last time:

How was guest lecture?

Reminders

- 2 things left:

- Request on evals

Quivers

A quiver is a directed graph:

Q_0 : vertices

Q_1 : edges (t_a, h_a), $a \in Q_1$

t_a & h_a : head & tail map

FIGURE A.1. Left: the quiver with $Q_0 = \{1,2,3\}$, $Q_1 = \{a,b,c,d,e\}$, $h : (a,b,c,d,e) \mapsto (2,2,3,1,1)$, $t : (a,b,c,d,e) \mapsto (1,2,2,3,3)$. Right: the underlying undirected graph.

Finite if:

A representation of Q over a field k is a pair $\mathbb{V} = (V_i, v_a)$ consisting of a set of k-vector spaces $\{V_i \mid i \in Q_0\}$ and a set of k-linear maps

$$\{v_a : V_{ta} \to V_{ha} \mid a \in Q_1\}$$

finite dimensional if

Example: Chain complexes

More abstract: a zigzag

$$\mathbb{Z}_2 \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \mathbb{Z}_2^2 \xleftarrow{\begin{pmatrix}1\\1\end{pmatrix}} \mathbb{Z}_2 \xleftarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \mathbb{Z}_2^2 \xrightarrow{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}} \mathbb{Z}_2^2$$

What is happening?

A morphism between 2 k-representations V & W of Q is a set of k-linear maps $\phi_i : V_i \to W_i$ such that the following commutes for all $a \in Q_1$:

$$V_{t_a} \xrightarrow{v_a} V_{h_a}$$

$$W_{t_a} \xrightarrow{w_a} W_{h_a}$$

(isomorphism if each ϕ_i is bijective)

Example: Inclusions of chain complexes $L \subseteq K$

$$
\begin{tikzcd}
C_n(L) \arrow[r, "\partial_n"] \arrow[d] & C_{n-1}(L) \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_1"] & C_0(L) \\
{} & & &
\end{tikzcd}
$$

More abstract:

$$\begin{tikzcd}
\boldsymbol{k} \arrow[r, "\binom{1}{0}"] \arrow[d, "\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[d, "0"'] & \boldsymbol{k} \arrow[l, "\binom{1}{1}"'] \arrow[d, "-\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[l, "\begin{pmatrix} 0 & 1 \end{pmatrix}"'] \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & -1 \end{pmatrix}"'] & \boldsymbol{k}^2 \arrow[d, "0"'] \\
\boldsymbol{k} \arrow[r, "0"] & 0 & \boldsymbol{k} \arrow[l, "0"'] & \boldsymbol{k} \arrow[l, "\mathbb{1}"'] \arrow[r, "0"] & 0
\end{tikzcd}$$

Check:

Quiver representations are like vector spaces:

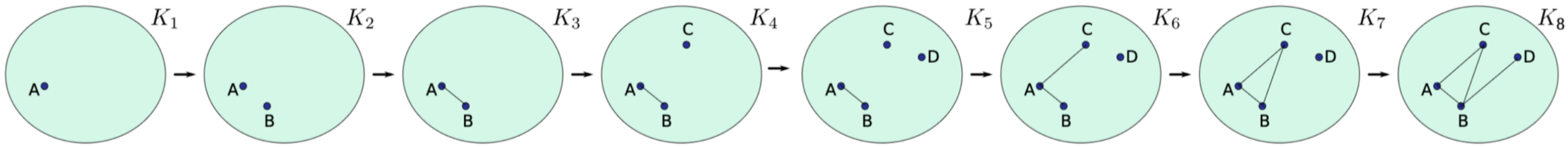
- They contain a 0 object
  ↳ all spaces & maps $= 0$
- They have a bi-product, the direct sum:
  $\mathbb{V} \oplus \mathbb{W}$: spaces $V_i \oplus W_i$, $i \in Q_0$
  maps $V_a \oplus W_a = \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix}$
- Every morphism $\phi: \mathbb{V} \to \mathbb{W}$ has a kernel: $(\ker \phi)_i = \ker \phi_i$
  (as well as image & cokernel)

A non-trivial representation $\mathbb{V}$ is called decomposable if it is isomorphic to the direct sum of 2 non-trivial representations.

(Otherwise indecomposable.)

Back to persistence for a minute...

Let $k = \mathbb{Z}_2$ & do H_0:



$$\begin{array}{ccccccccccccccc}
H_0(K_1) & \longrightarrow & H_0(K_2) & \longrightarrow & H_0(K_3) & \longrightarrow & H_0(K_4) & \longrightarrow & H_0(K_5) & \longrightarrow & H_0(K_6) & \longrightarrow & H_0(K_7) & \longrightarrow & H_0(K_8) \\
\cong & & \cong & & \cong & & \cong & & \cong & & & & & & \\
\mathbb{Z}_2 & \to & \mathbb{Z}_2^2 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_2^2 & \to & \mathbb{Z}_2^3 & & \mathbb{Z}_2^2 & & \mathbb{Z}_2^2 & & \mathbb{Z}_2
\end{array}$$

$$\underbrace{\mathbb{Z}_2 \to \mathbb{Z}_2^2}_{\begin{bmatrix}1\\0\end{bmatrix}} \underbrace{\to \mathbb{Z}_2}_{\begin{bmatrix}1 & 0\end{bmatrix}}$$

decompose:

We say $\mathbb{W} = (W_i, w_a)$ is a subrepresentation of $\mathbb{V} = (V_i, V_a)$ if

- W_i is a subspace of V_i $\forall i$
- and w_a is the restriction of map V_a to W_i, $\forall a \in Q_1$

Example:

$$\text{Cycles } Z_n = \ker \partial_n$$
$$\subseteq$$

Central question

Classify representations of a given quiver up to isomorphism.

[usually all finite + finite dim]

Define $\underline{\dim}\, \mathbb{V} = (\dim V_1, \ldots, \dim V_n)^T$

a vector

and $\dim \mathbb{V} =$

How to get a handle on this?

Krull-Remak-Schmidt Theorem

$\boxed{\text{Wedderburn 1909, Remak 1911, Schmidt 1913, Krull 1925}}$

Assuming Q is finite, for any $\mathbb{V} \in \operatorname{rep}_k(Q)$, $\exists$ indecomposable representations $\mathbb{V}_1, \dots \mathbb{V}_r$ s.t.

$$\mathbb{V} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r.$$

Moreover, for any other indecomposable rep. $\mathbb{W}_1, \dots, \mathbb{W}_s$ with $\mathbb{V} = \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_s$ must have $r = s$ and the W_i's + V_j's are permutations.

So → to classify, need to understand & characterize indecomposables.

$\boxed{\text{Gabriel's theorem 1972}}$

Let Q be finite quiver & k a field. Then, Q has a finite # of classes of indecomposables $\iff$ Q is Dynkin.

Why surprising?

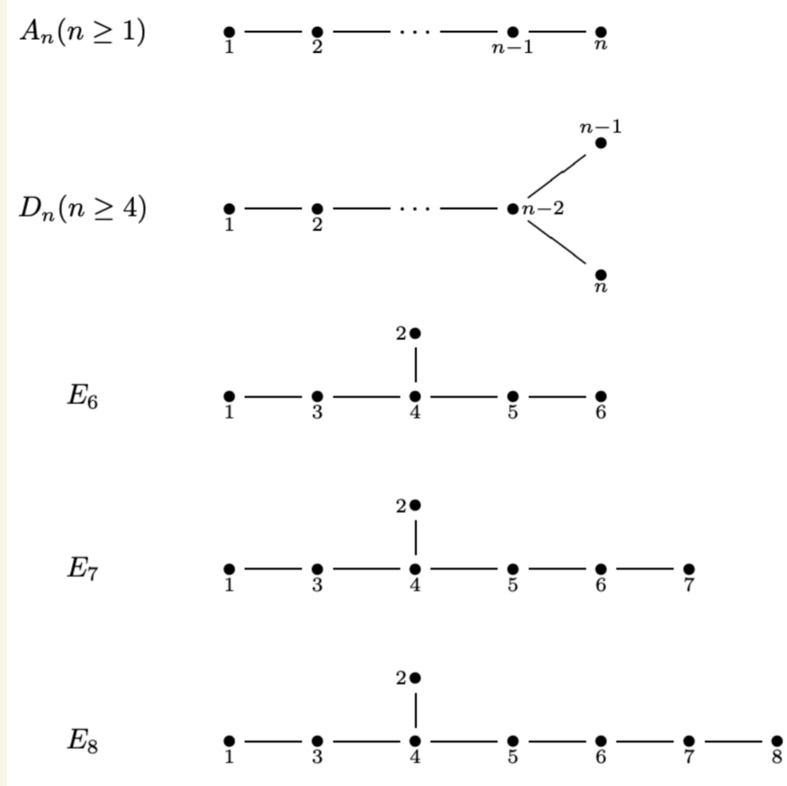


FIGURE A.2. The Dynkin diagrams.

Second part of Gabriel's work identifies the indecomposables of the Dynkin quivers with elements in root systems of polynomials.

(If curious: quadratic form called "Tit's form", & Dynkin $\iff$ form is positive definite.)

Back to persistence

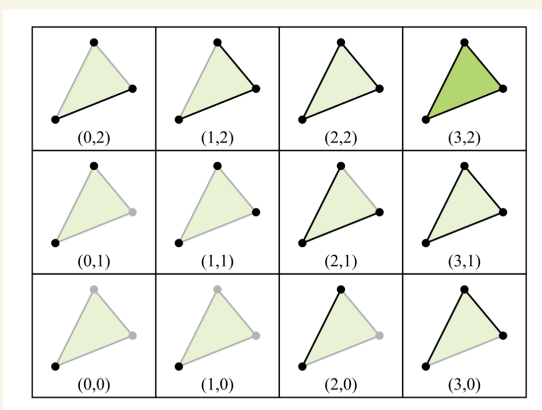
Some limitations here!

- Only finitely indexed set quivers, & many filtrations indexed over $\mathbb{R}$

But, luckily later theory addresses this.

(And, in practice, computers are finite!)

Why we still care?



$\Longrightarrow$

$$
\begin{tikzcd}
k^2 \arrow[r] & k \arrow[r] & k \arrow[r] & k \\
k^2 \arrow[r] \arrow[u] & k^3 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u] \\
k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}
$$

Interestingly, can also extend most of this to arbitrary posets:

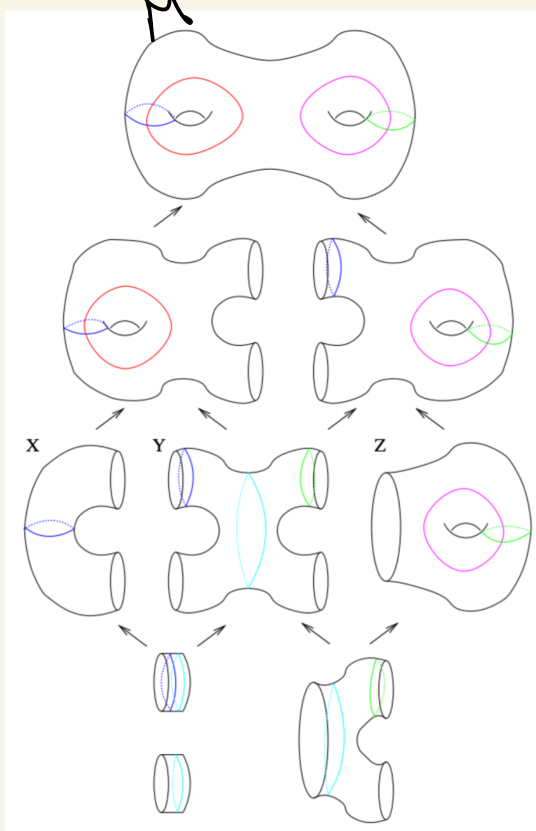
$$
\begin{tikzcd}
X_1 \arrow[rr] \arrow[dr] & & X_5 \\
& X_3 \arrow[r] \arrow[dr] & X_6 \\
X_2 \arrow[ur] \arrow[r] & X_4 \arrow[r] \arrow[ur] & X_7
\end{tikzcd}
\qquad
\begin{tikzcd}
H_k(X_1) \arrow[rr] \arrow[dr] & & H_k(X_5) \\
& H_k(X_3) \arrow[r] \arrow[dr] & H_k(X_6) \\
H_k(X_2) \arrow[ur] \arrow[r] & H_k(X_4) \arrow[r] \arrow[ur] & H_k(X_7)
\end{tikzcd}
$$

$\mathbb{Z}_2$ H_1 representation:

Example:

$M =$ $X \cup Y \cup Z$ & take intersections

M



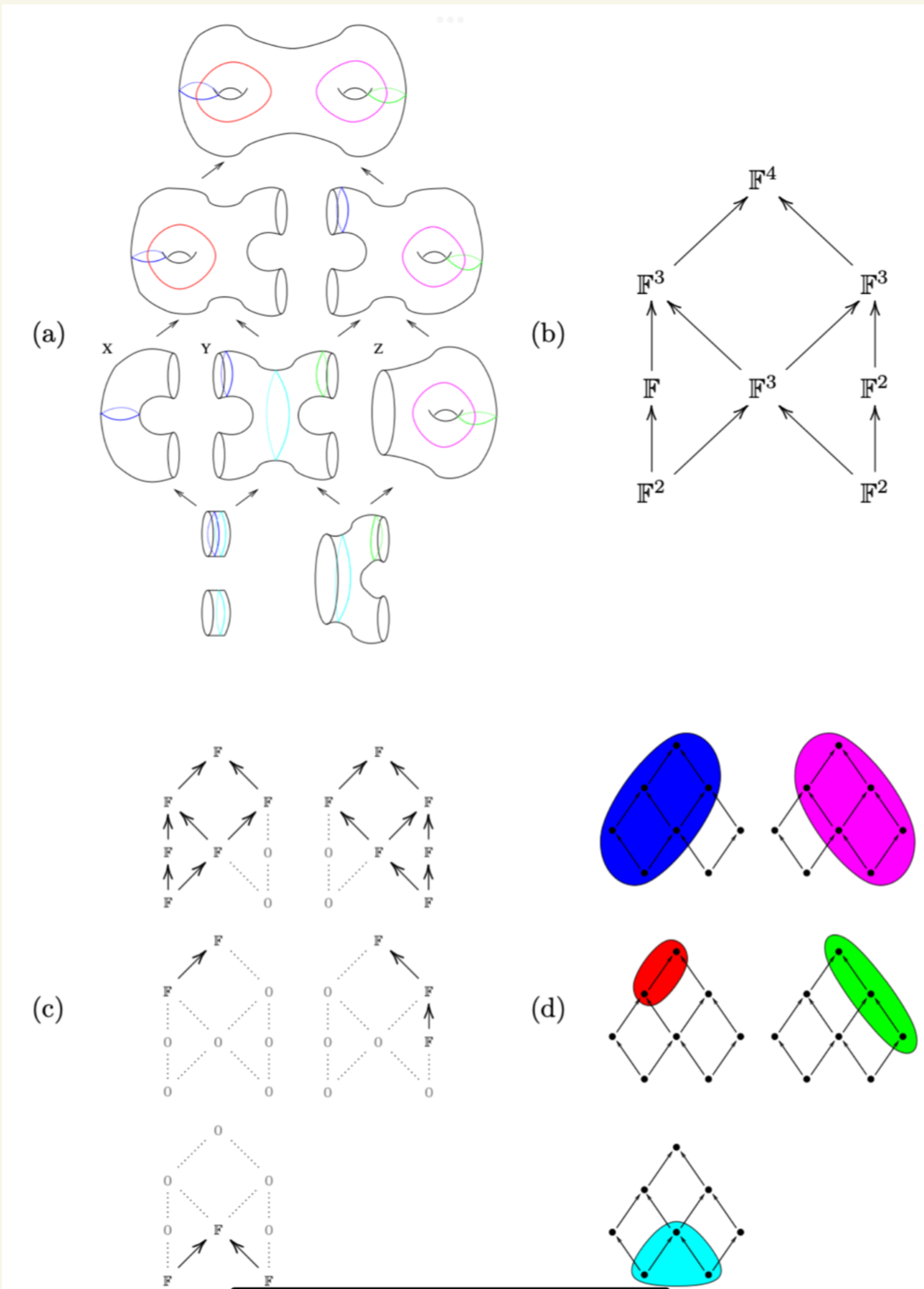
But, since <u>not</u> Dynkin in general, Gabriel's theorem doesn't apply
$\Rightarrow Q$ has infinite # of isomorphism classes of indecomposables.

Translating:

Invariants

Carrier subgraph:

C.-Letscher 2019



Biparameter filtrations again

① Dimension Function

Simply map each $a \in \mathbb{R}^2$ to $\dim(M_a)$:

$$\begin{tikzcd}
k^2 \arrow[r] & k \arrow[r] & k \arrow[r] & k \\
k^2 \arrow[r] \arrow[u] & k^3 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u] \\
k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd} \implies$$

pros:

cons:

② Rank invariant: first an example

Let $P = \{0,1,2\} \times \{0,1,2\} \subset \mathbb{Z}^2$

with usual partial order: P:

$(i,j) \leq (i',j')$
$\iff i \leq i'$ & $j \leq j'$

Let's build a bipersistent module as direct sum of 2 rectangles:

$R_A = \{(i,j) \mid i \in \{0,1,2\}, j \in \{0,1\}\}$

$R_B = \{(i,j) \mid i \in \{1,2\}, j \in \{0,1,2\}\}$

Example continued:

For each $p=(i,j)$,

$$A(p) = \begin{cases} k & \text{if } p \in R_A \\ 0 & \text{if not} \end{cases}$$

$$B(p) = \begin{cases} k & \text{if } p \in R_B \\ 0 & \text{if not} \end{cases}$$

& all maps either 0 or 1

Let $M = A \oplus B$

dimension grid here:

$$\begin{matrix} 0 & k & k \\ k & k^2 & k^2 \\ k & k^2 & k^2 \end{matrix} \implies$$

Rank invariant: $\forall\, p \leq q$, defined as

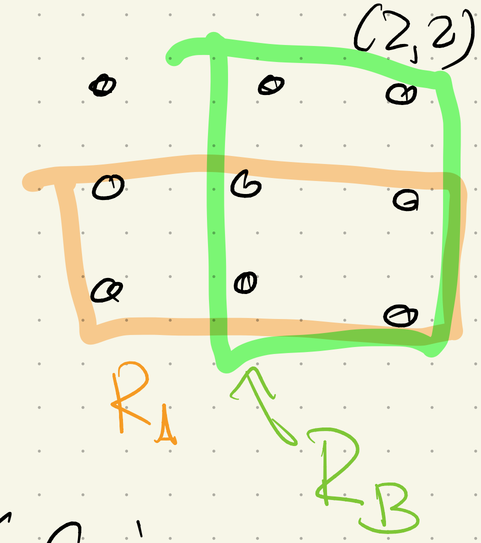
$$\mathrm{rank}_M(p,q) = \mathrm{rank}(p \to q)$$

~~Here~~, $\mathrm{rank}(p \to q) =$

So, trying to compute:

fix $q = (2,2)$

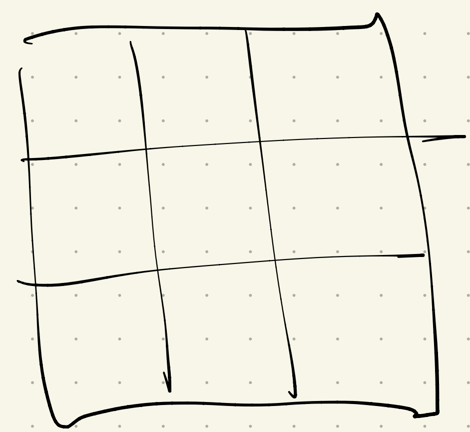
$q \notin R_A$, but $q \in R_B$



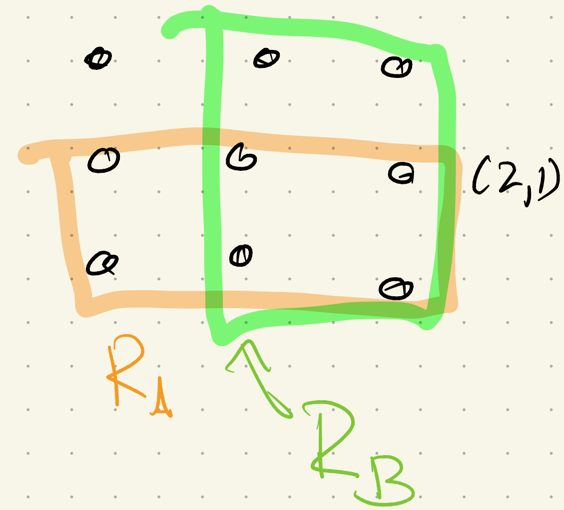
Then consider all p s.t. $p \leq q$:

$$\mathrm{rank}(p \to q) = \underbrace{\mathrm{rank}_A(p \to q)}_{\text{here:}} + \mathrm{rank}_B(p \to q)$$

so: $\mathrm{rank}_q(p) =$

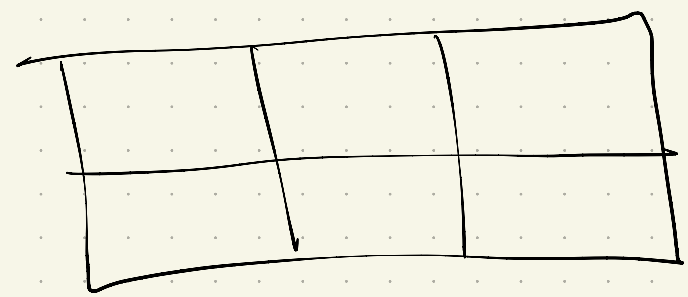


Another: fix $q = (2,1)$



Now, all $p \le q$!

$$\mathrm{rank}_q(p) =$$



$$= \mathrm{rank}_A(p \to q) + \mathrm{rank}_B(p \to q)$$

This still (in a sense) measures
"homological features in p that
persist until q"

But: non-isomorphic modules can
share rank invariants
& can't have "good barcodes"

Let's unpack why...

"Good" barcodes: what we mean

Consider a multiset of subsets of $P \rightsquigarrow$ a barcode

in standard: $M_1 \to M_2 \to \cdots \to M_n$

in multi-d:

$$\begin{array}{ccccccc}
M_{n,1} & \to & M_{n,2} & \to & \cdots & \to & M_{n,n} \\
\uparrow \vdots & & \uparrow \vdots & & & & \uparrow \vdots \\
M_{2,1} & \to & M_{2,2} & \to & \cdots & \to & M_{2,n} \\
\uparrow & & \uparrow & & & & \uparrow \\
M_{1,1} & \to & M_{1,2} & \to & \cdots & \to & M_{1,n}
\end{array}$$

Say it is good if $\forall\, x \le y$,

$$\mathrm{Rank}(M_x \to M_y) = \left|\{S \in B \mid x, y \in S\}\right|$$

↳ # elements w/ x & y

2d modules are <u>not</u> good:

$\boxed{\text{Botnan-Lesnick 2023}}$

$$\begin{tikzcd}
k \arrow[r, "="] & k \arrow[r] & 0 \\
k \arrow[u, "="] \arrow[r, "{[1,0]^T}"] & k^2 \arrow[u, "{[1,0]}"'] \arrow[r, "{[1,1]}"] & k \arrow[u] \\
0 \arrow[u] \arrow[r] & k \arrow[u, "{[0,1]^T}"'] \arrow[r, "="] & k \arrow[u, "="']
\end{tikzcd}$$

Suppose B is a good barcode:

Know:

$$\mathrm{Rank}(M_{(0,1)} \to M_{(2,1)}) = \mathrm{Rank}(M_{(0,1)} \to M_{(1,2)})$$
$$= \mathrm{Rank}(M_{(1,0)} \to M_{(2,1)}) =$$

What must B contain?

$$
\begin{tikzcd}
k \arrow[r, "="] & k \arrow[r] & 0 \\
k \arrow[u, "="] \arrow[r, "{[1,0]^T}"] & k^2 \arrow[u, "{[1,0]}"'] \arrow[r, "{[1,1]}"] & k \arrow[u] \\
0 \arrow[u] \arrow[r] & k \arrow[u, "{[0,1]^T}"'] \arrow[r, "="] & k \arrow[u, "="']
\end{tikzcd}
$$

Need

$(0,1), (2,1) \in I$

$(0,1), (1,2) \in J$

$(1,0), (2,1) \in K$

& $I, J, K \in B$

But $\dim M_{0,1} = \dim M_{2,1} = 1$

$\Longrightarrow$

Contradiction!

What is $\mathrm{Rank}(M_{(0,1)} \to M_{(1,2)})$?

Note: This is a module whose rank invariant is not equal to the rank invariant of any interval-decomposable module.

But: is the difference between rank invariants of 2 interval decomposable modules!

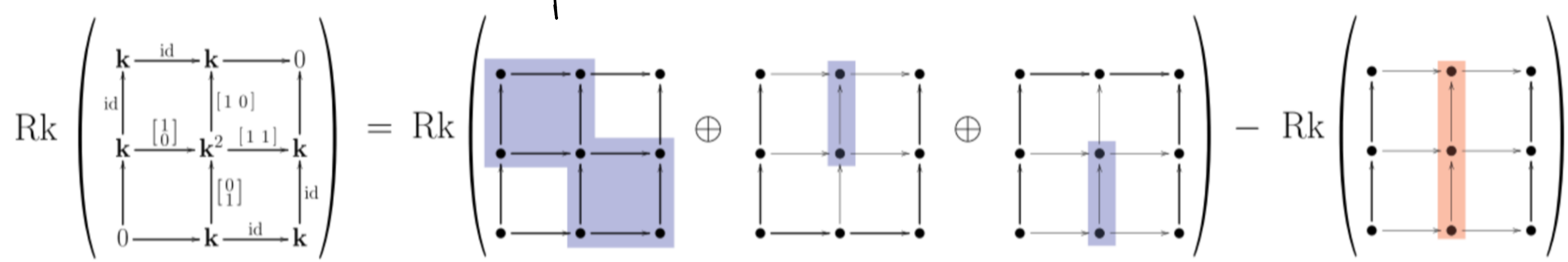
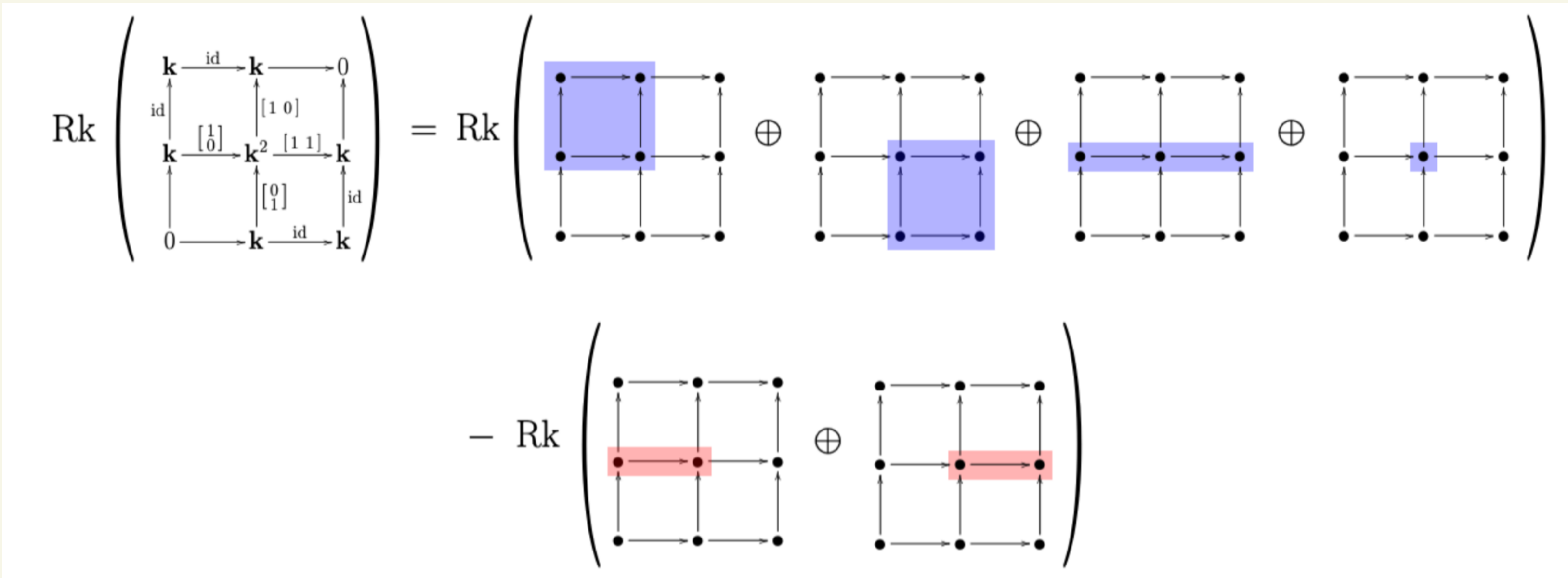


Fig. 2 The indecomposable module M on the left-hand side does not have the same rank invariant as any direct sum of interval modules on the 3×3 grid. However, Rk M is equal to the difference between the rank invariants of two direct sums of interval modules, as shown on the right-hand side. Blue is for intervals counted positively in the decomposition, while red is for intervals counted negatively (Color figure online)

This can be useful

↳ but not unique:



New trends connect this to something called the Möbius inversion.

Patel 2018

Patel & Skraba '23