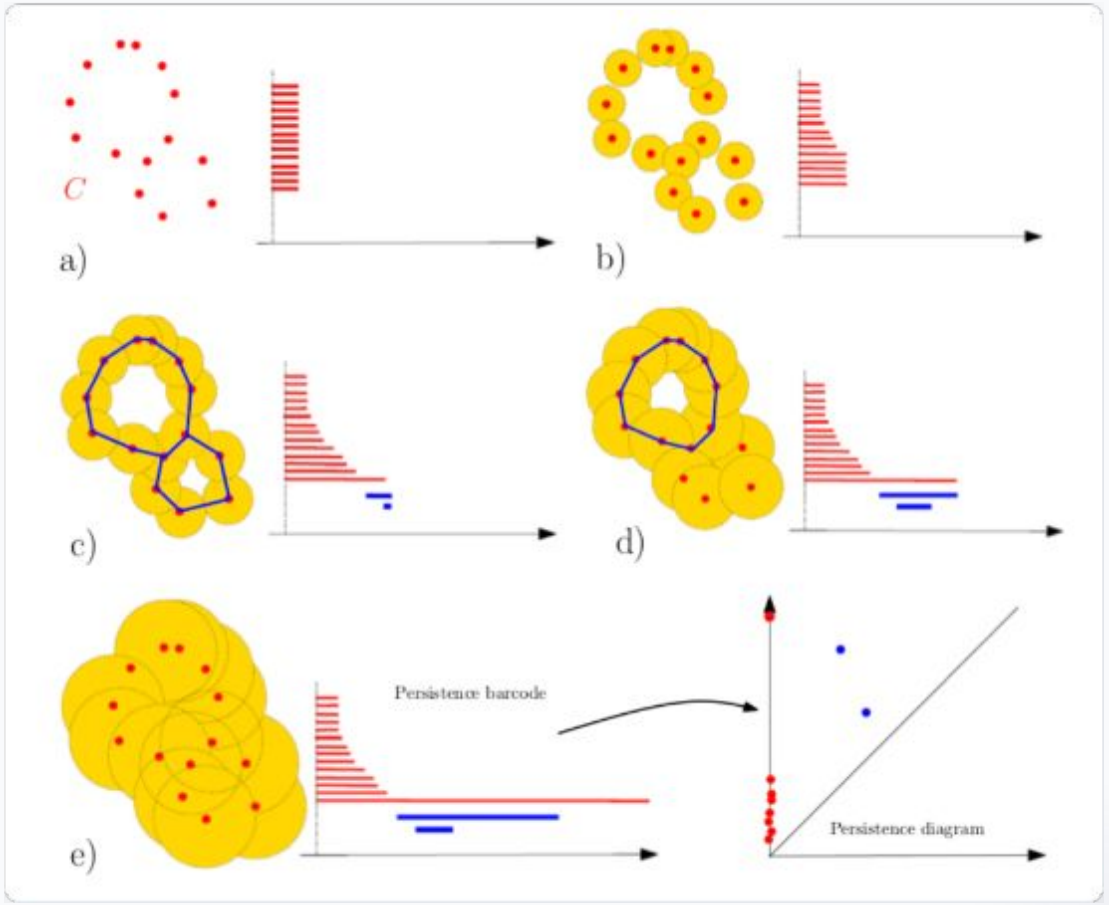
Local Detection of Monodromy

Shankha Shubhra Mukherjee

Algorithmic Explorations of Symmetry Sets and Evolutes

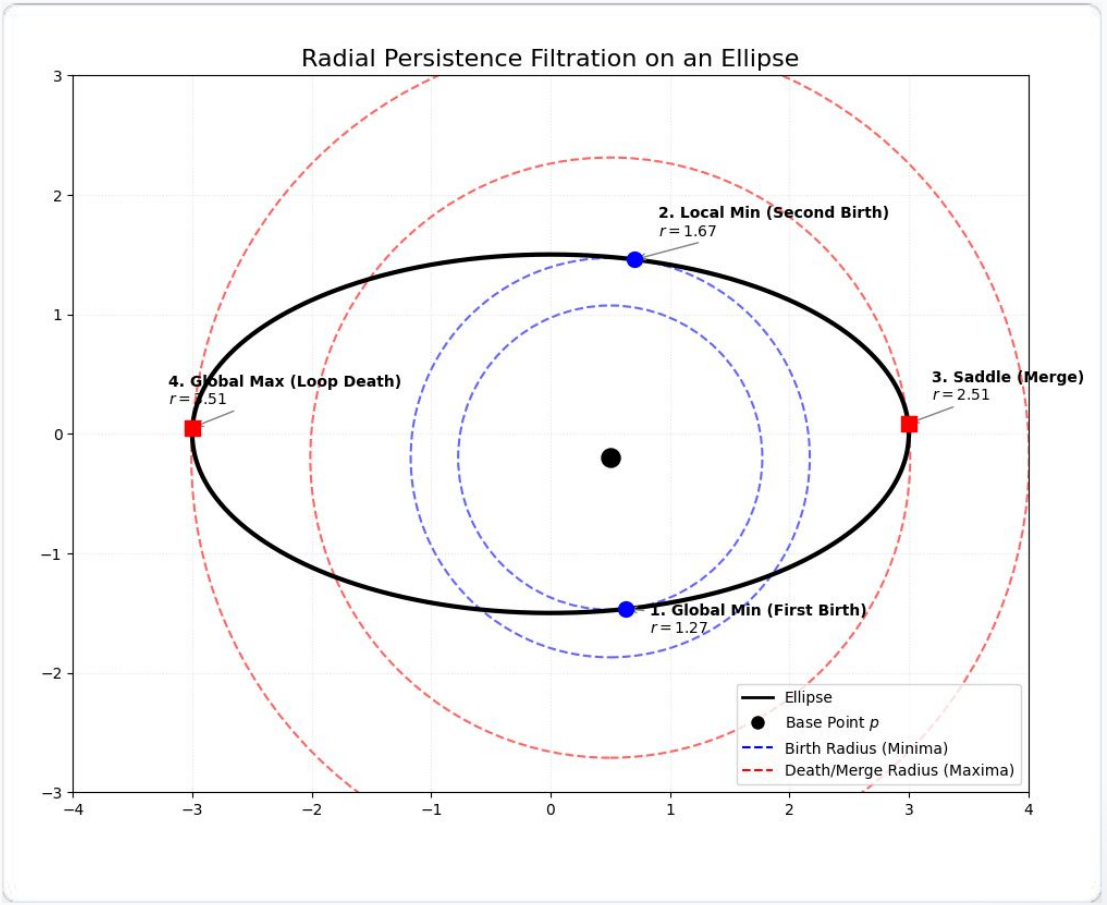
Persistent Homology

- **Definition:** A method for computing topological features (components, holes, voids) of a space at different spatial resolutions.
- **Filtration:** Tracks the evolution of homology groups as a scale parameter increases.
- **Robustness:** Extracts quantitative invariants from complex, noisy data.
- **Output:** Typically represented as a Barcode or Persistence Diagram.



Radial Persistence Homology

- **Key Difference:** Uses a *Distance-to-Point* filtration instead of a global height or density function.
- **Parameter Space:** The filtration depends on a base point $p \in R^d$.
- **Dynamic:** As p moves, the persistence diagram changes continuously.
- **Perspective:** Captures the "view" of the shape's topology from a specific location.



What is Monodromy?

The Concept

Monodromy describes how a mathematical object changes as you move around a singularity and return to the start.

Specifically, it tracks the non-trivial permutation of features after traversing a closed loop in parameter space.

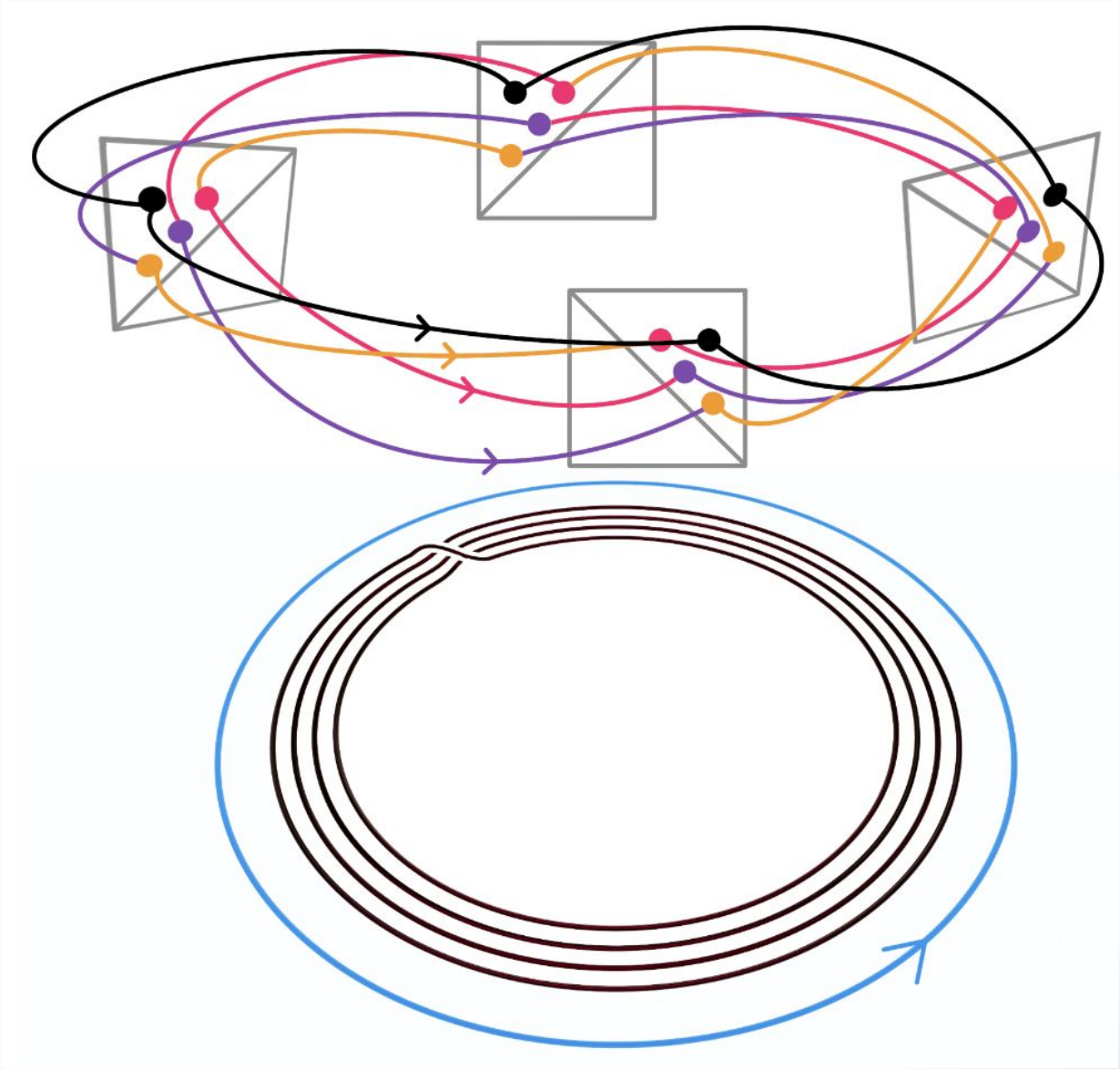
Visual Intuition

Imagine holding a bundle of strings. As you walk in a circle, the strings might twist or braid around each other.

If you return to the start and the strings are swapped, that is **Monodromy**.

Monodromy in Radial Persistence

- **The Bundle:** The stack of persistence diagrams over the parameter space
- **The Action:** Moving the base point p in a loop γ
- **The Effect:** Topological features (points in the diagram) trace paths called "vines."
- **Detection:** If features A and B swap labels after the loop, we have detected a singularity inside the loop.



Why this matters

Feature Stability

Monodromy acts as an obstruction to globally coherent labeling. Detecting it helps regularize unstable medial structures.

Pruning Strategy

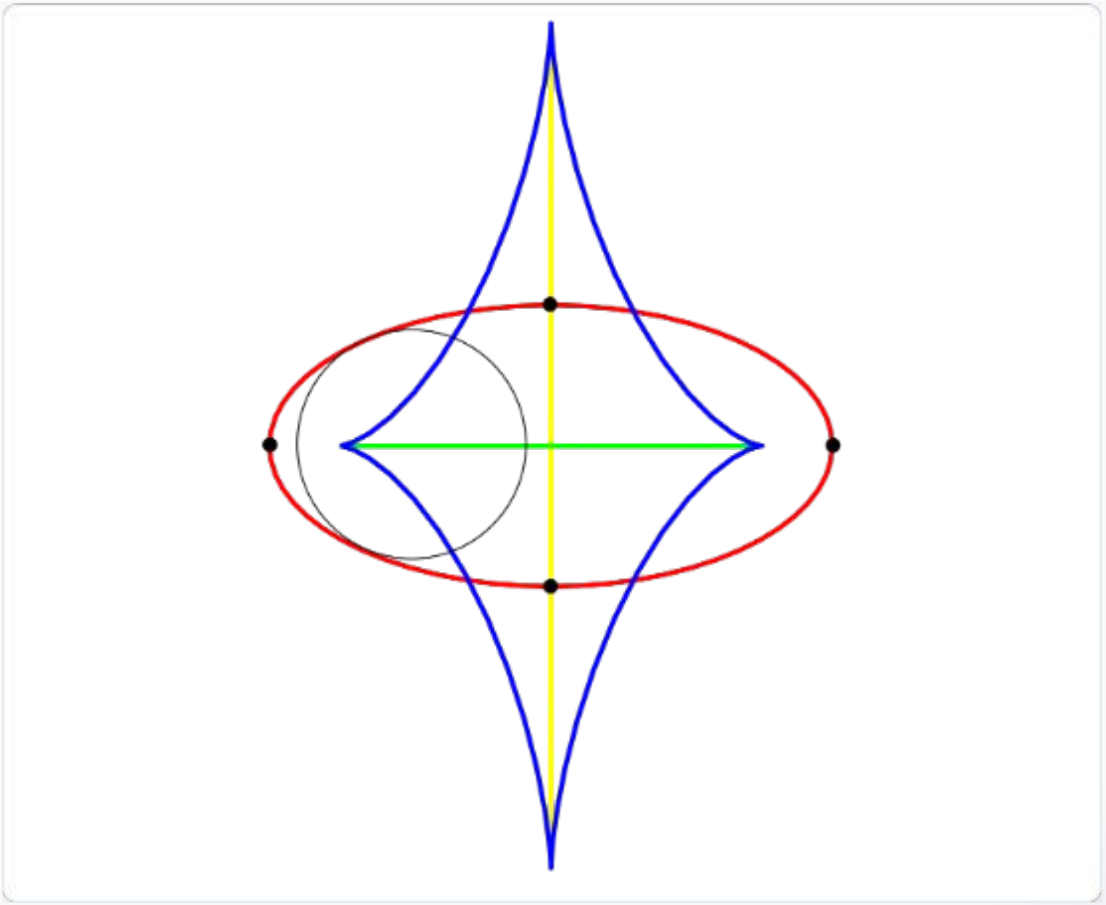
Branches with high monodromy but low persistence can be systematically pruned to denoise skeletal descriptors.

Shape Invariants

Braid-type invariants from vineyards provide new numerical signatures for distinguishing shape configurations.

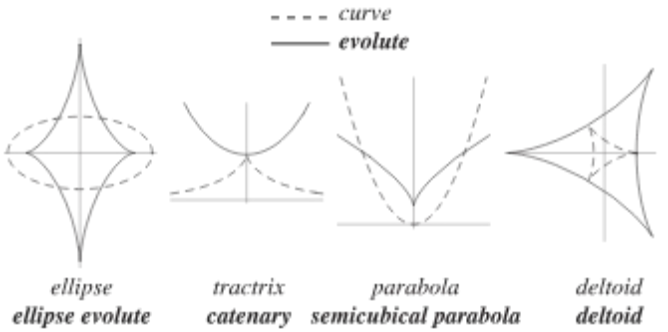
The Symmetry Set

- **Definition:** The closure of the set of points p where the ball centered at p is tangent to the manifold at 2 points.
- **Medial Axis:** A subset of the symmetry set (centers of maximal empty balls).
- **Role:** These are the "singularities" inside the shape that cause the topological shifts.



The Focal Set

- **Definition:** The locus of centers of curvature of the manifold (Evolute in 2D).
- **Singularity:** Points where the distance function has degenerate critical points.
- **Connection:** Crossings of symmetry branches often align with focal set cusps.



Implementation: Setup & Filtration

1. Shape Definition

We model the shape S as a point cloud or dense boundary mesh.

Symmetry Breaking: We use irregular shapes (e.g., "Bean", "R-shape") to ensure generic intersections of symmetry branches.

2. Radial Filtration

For every parameter point p in a dense grid:

- Compute distance $f_p(x) = |x - p|$.
- Construct the Rips filtration.
- Compute Persistence Diagram using GUDHI.

Implementation: Detection Logic

3. Optimal Matching

We use the **Hungarian Algorithm** (Wasserstein distance) to map features between adjacent grid points.

This creates a discrete bundle of persistence modules.

4. The Loop Check

For every elementary grid square:

- Compose matchings:

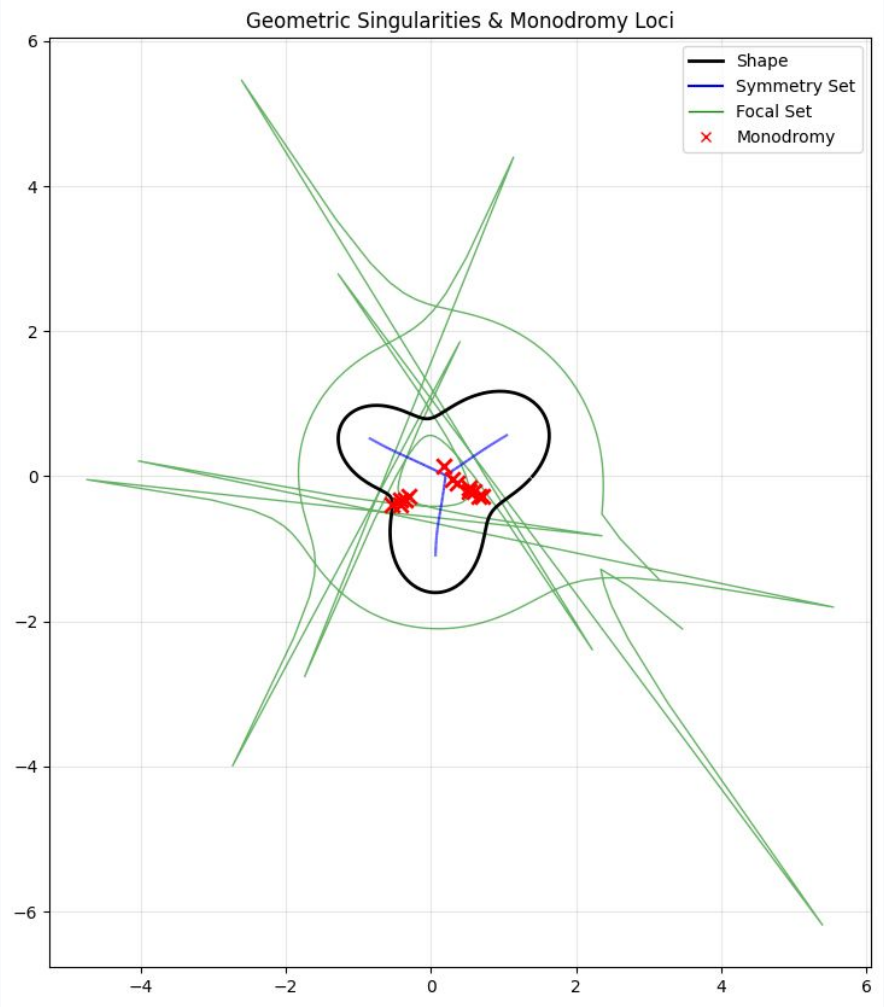
$$M_{total} = M_{4\to1} \dots M_{1\to2}$$

- Check if M_{total} is the Identity map.
- **Non-Identity** indicates non trivial Monodromy has been detected.

Output: Geometric Structures

Computed Sets

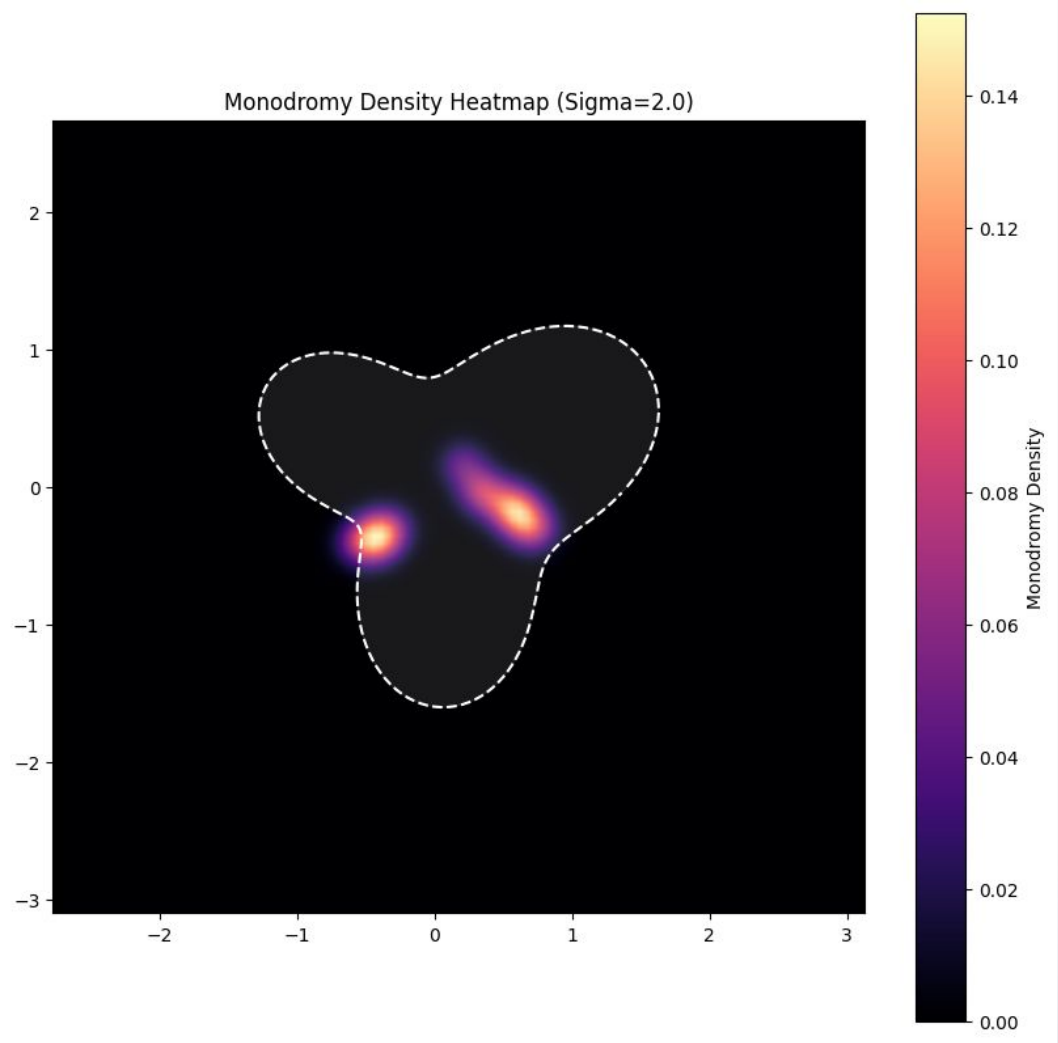
- **Black:** Shape Boundary.
- **Blue:** Symmetry Set (Medial Axis) via Voronoi filtering.
- **Green:** Focal Set (Evolute) via discrete curvature.
- **Red Crosses:** Discrete locations where braiding was detected.



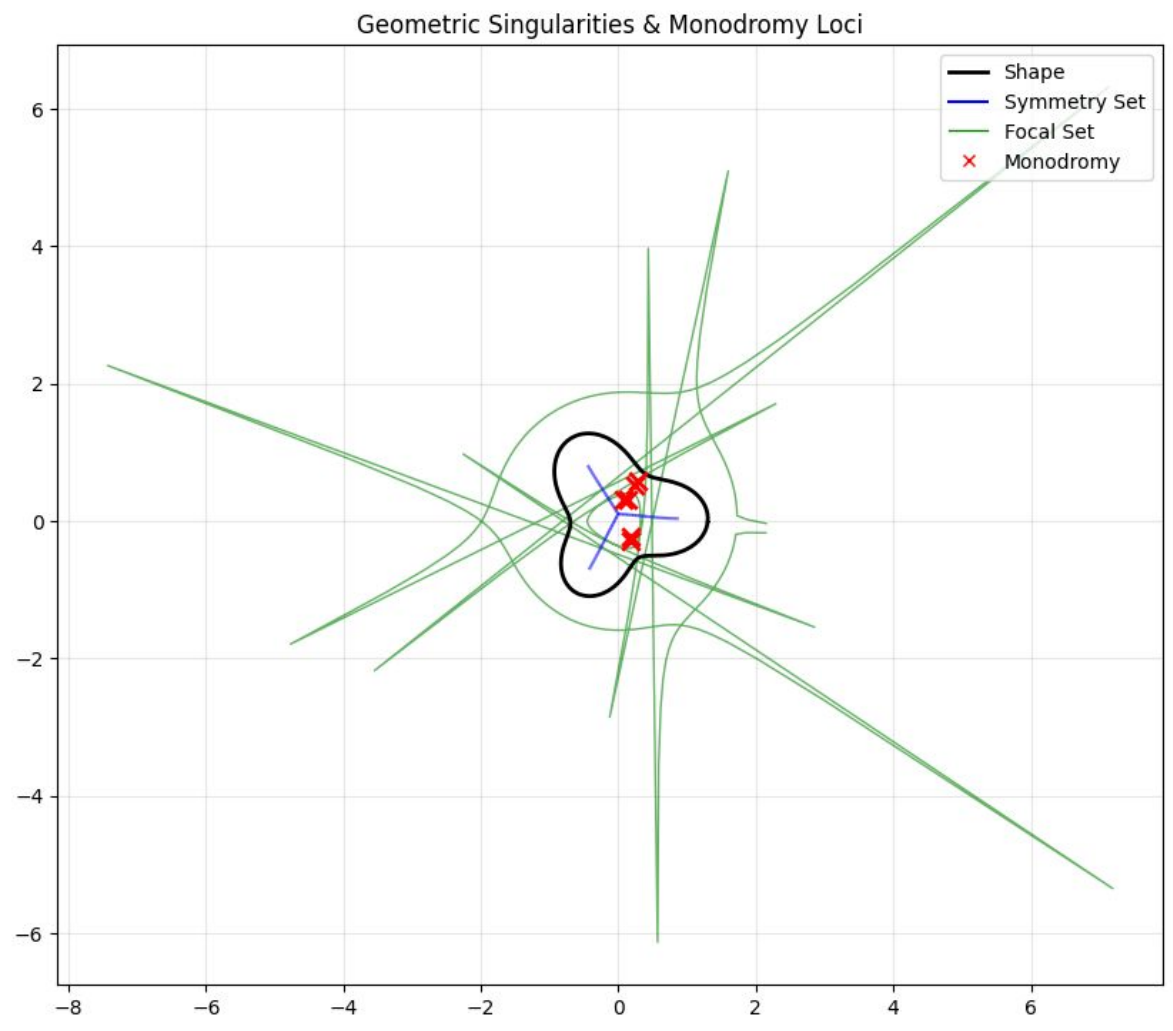
Output: Monodromy Heatmap

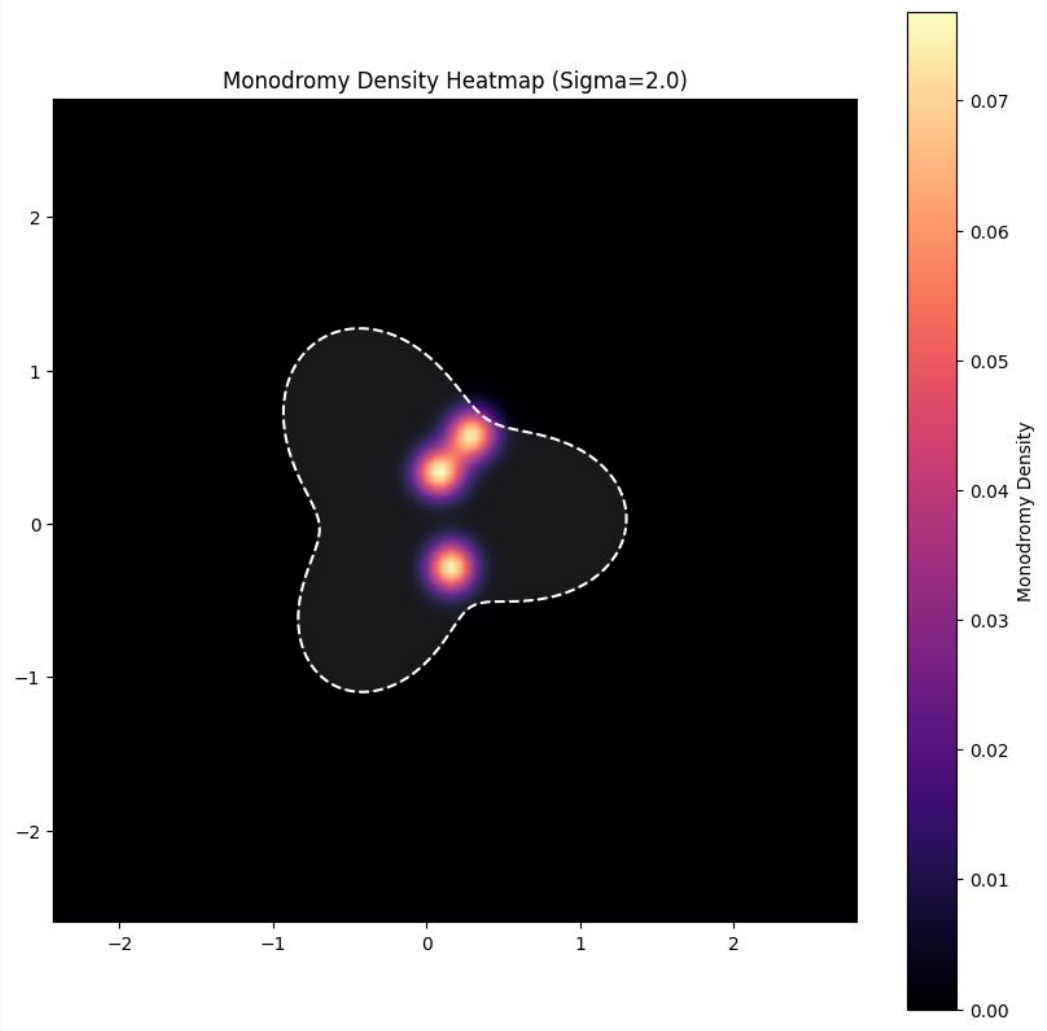
Detection Results

- **Heatmap:** Density of monodromy events.

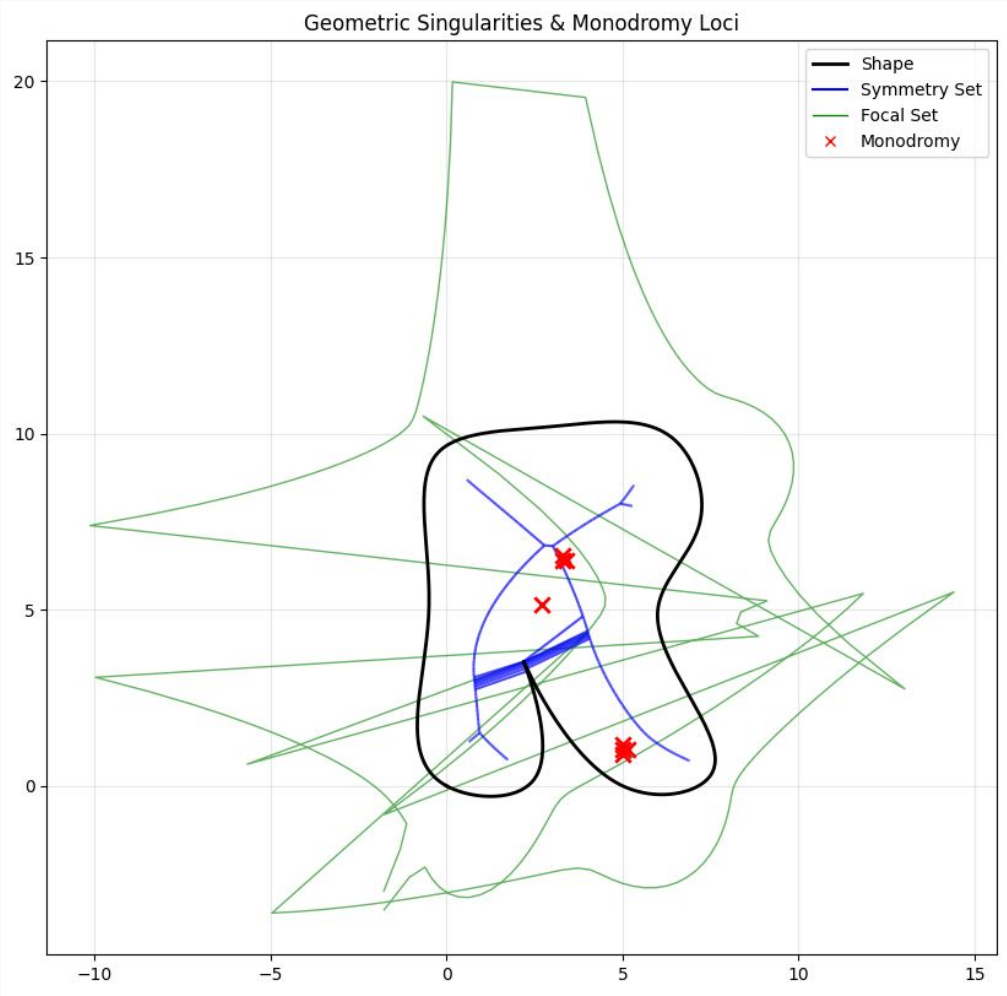


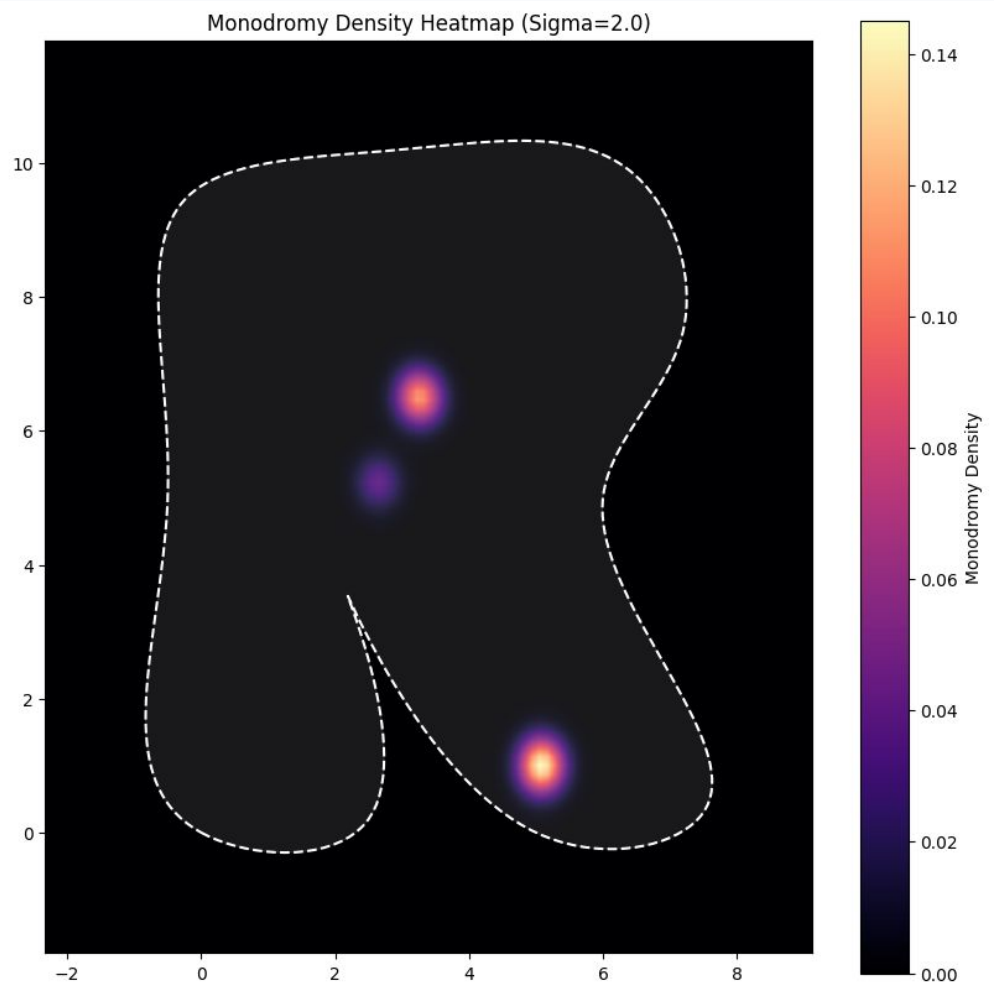
Output: Irregular Trefoil(variation)





Output: R shape





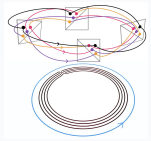
Future Work

- **Guided Pruning:** Use monodromy intensity to automatically prune unstable medial axis branches.
- **Higher Dimensions:** Extend the detector to 3D shapes and surface evolutes.
- **Braid Invariants:** Extract full braid words (Artin generators) to classify singularity types.

Image Sources

https://raw.githubusercontent.com/GUDHI/TDA-tutorial/71e415c8680e5255eecb3e51393a51c04a15f402/Images/persistence.png

Source: github.com



https://arxiv.org/abs/2504.11203

Source: Erin Chambers, Christopher Fillmore, Elizabeth Stephenson, Mathijs Wintraecken. Braiding Vineyards, 2025

https://upload.wikimedia.org/wikipedia/commons/thumb/e/e9/Ellipse_symmetry_set.svg/1200px-Ellipse_symmetry_set.svg.png

Source: en.wikipedia.org

https://mathworld.wolfram.com/images/eps-svg/Evolutes_951.svg

Source: mathworld.wolfram.com

Thank You!