Vectorizing Locally Trivial Persistence Diagram Bundles

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Vineyards

Vineyards, originally introduced in [CEM06], are time-varying persistence diagrams associated to time-varying filtrations. One visualize the time variance as a “stack" of PDs, and using this viewpoint can examine the “vines" traced out by points on PDs moving through time, hence the name “vineyard". These vines correspond to persistent homology classes which “persist" through some interval of time variance of the filtration. This construct loosely resembles something known as a "fiber bundle", a mathematical object which consists of a base space B, and "fibers" over each point in B homeomorphic to a fixed topological space F. Specifically, a vineyard is somewhat like a fiber bundle over $\mathbb{R}$ where each fiber is a persistence diagram. Until recently, there was not a general notion of a persistence diagram fibered bundle.

The Persistent Homology Transform

One motivating example for generalizing the base space of vineyards from $\mathbb{R}$ to arbitrary topological spaces is the Persistent Homology Transform. Let $M \subset \mathbb{R}^d$ be a geometric simplicial complex, and let $\mathcal{D}$ denote the space of persistence diagrams. For each unit vector $v \in S^{d-1}$ there is an associated sublevel filtration

$$M(v)_r = \{x \in M | \langle x, v\rangle \leq r\}$$

We denote the persistence diagram of this filtration in degree k homology by $X_k(M, v)$. The Persistent Homology Transform of M is then the map $\mathrm{PHT}(M) : S^{d-1} \to \mathcal{D}^d$ defined by

$$v \mapsto (X_0(M, v), \dots, X_{d-1}(M, v))$$

Each graded component of the PHT is like a bundle of persistence diagrams over S^{d-1}.

PD-bundles

A persistence diagram bundle is a pair (B, f) where B is a base space and $f : B \to PD$ is a function continuous with respect to the bottleneck metric. The total space of a persistence diagram is defined to be $E := \{(b, p) | b \in B, p \in f(b)\}$. Frequently, we use (B, f) to refer to both the entire structure of the PD-bundle and the total space of the PD-bundle interchangeably for the sake of convenience. Persistence diagram bundles always rest in a specific homology degree.

Sections

A section of a PD-bundle is a map $s : B \to E$ such that $s(b) \in f(b)$ and the set $\{b \in B | s(b) \notin \Delta\}$ is connected.

Monodromy

A PD-bundle (B, f) is said to have trivial geometric monodromy if it has a family of sections $\{\gamma_i : B \to \overline{\mathbb{R}}^2\}$ such that $f(p) \backslash \Delta = \bigcup_{\{i | \gamma_i(p) \notin \Delta\}} \gamma_i(p)$. Essentially, this means that the PD-bundle is trivial.

Vector Bundles

The most well understood class of "bundle" is the vector bundle. For each element of the base $x \in B$, the fiber of the vector bundle E above x is a vector space. This ensemble of fibers is called a vector bundle, provided that the total space $E := \{(x, v) | x \in B, v \in E_x\}$ has a topology which is locally trivial, meaning there exists an open cover of B, $\{\mathcal{U}_i\}_{i \in I}$, such that $E|_{\mathcal{U}_i} \cong \mathcal{U}_i \times \mathbb{R}^n$ for some fixed n.

Initial Project Goal

Goal: Turn problems related to monodromy of a certain class of PD-bundles into problems related to the monodromy of vector bundles.

Vectorization

Let (B, f) be a PD-bundle over B determined by a map $f : B \to PD$. We say that (B, f) is locally trivial if there exists an open covering of B, $\{\mathcal{U}_i\}_{i \in I}$ such that $(B, f)|_{\mathcal{U}_i}$ admits a finite number of fiberwise mutually distinct sections $s_r : \mathcal{U}_i \to (B, f)|_{\mathcal{U}_i}$ satisfying $f(p) \backslash \Delta = \bigcup s_r(p)$. We define the vectorization of (B, f), $V(B, f)$ to be the collection of fibers over B given by $V(B, f)_x := \mathrm{span}_{\mathbb{R}}\{p \in f(x)\} \oplus \Delta$. It turns out that this collection of fibers has the structure of a vector bundle.

The Obstruction Result

Theorem: Let (B, f) be a locally trivial PD-bundle. If (B, f) has trivial geometric monodromy, then $V(B, f)$ is a trivial vector bundle over B. What immmediately follows is that obstructions to the triviality of $V(B, f)$ obstruct trivial geometric monodromy in (B, f).

Stiefel-Whitney Classes

Stiefel-Whitney classes are algebraic invariants of vector bundles which lie in the $\mathbb{F}_2$-cohomology ring of the base space and obstruct the vector bundle from being trivial. Specifically, for $\xi, \xi^{'} \in \mathrm{Vect}_{\mathbb{R}}(X)$, and a connected base X Steifel-Whitney classes $\omega_i : \mathrm{Vect}_{\mathbb{R}}(X) \to H^i(X; \mathbb{F}_2)$ can be characterized by the following four axioms:

Axiom 1: Dimensionality

$\omega_0(\xi) = 1$, and $\omega_i(\xi) = 0$ for $i > \dim \xi$

Axiom 2: Naturality

$\omega_i(f^*\xi) = f^*\omega_i(\xi)$ for $f : Y \to X$

Axion 3: Whitney Formula

$$\omega_k(\xi \oplus \xi^{'}) = \sum_{i+j=k} \omega_i(\xi)\omega_j(\xi^{'})$$

Axiom 4: Normalization

Let γ_1^1 be the Mobius line bundle over S^1, then $\omega_1(\gamma_1^1) = 1 \in H^1(S^1; \mathbb{F}_2) \cong \mathbb{F}_2$

Future Directions

It is not currently known whether a converse to the obstruction result. The vectorization operation presented in this manuscript is limited to the case of persistence diagram bundles which possess a local triviality condition designed so that they resemble vector bundles more closely. This excludes PD-bundles which exhibit certain sorts of "interesting" behavior such as merging classes. One future research direction would be to describe a type of vectorization operation for a broader class of PD-bundles wherein the obstruction result still holds.

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References

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