TDA- fall 2025

Persistence

Recap

- HW2 - due in 1 week
- Next week: special assignment
  Intro: the AARTN
- Next week: no class
  (at a workshop)

And back to persistence...

Induced maps on homology

Each $K_i \hookrightarrow K_{i+1}$, so we get induced maps $H_p(K_i) \to H_p(K_{i+1})$.

Homology module (simplicial case):

$H_p(\mathcal{F}(K))$:

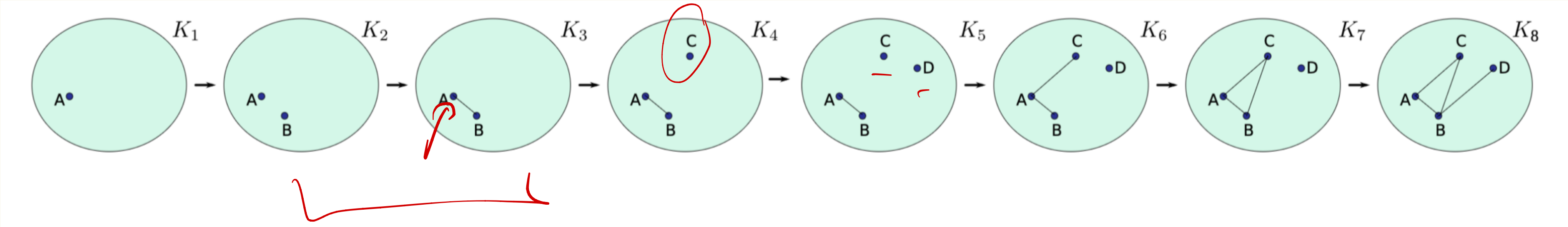
$$\begin{tikzcd}
\emptyset = H_p(K_0) \arrow[r] & \underset{i}{H_p(K_1)} \arrow[r] \arrow[rr, bend left, "\text{inclusion}"] & \cdots \arrow[r] & \overset{j}{H_p(K_n)} = H_p(K)
\end{tikzcd}$$

$i, j \to f_p^{i,j} : H_p(K_i) \to H_p(K_j)$

What do these capture?

$K_i \hookrightarrow K_j$, if $i < j$

Now let's try tracking generators of homology!

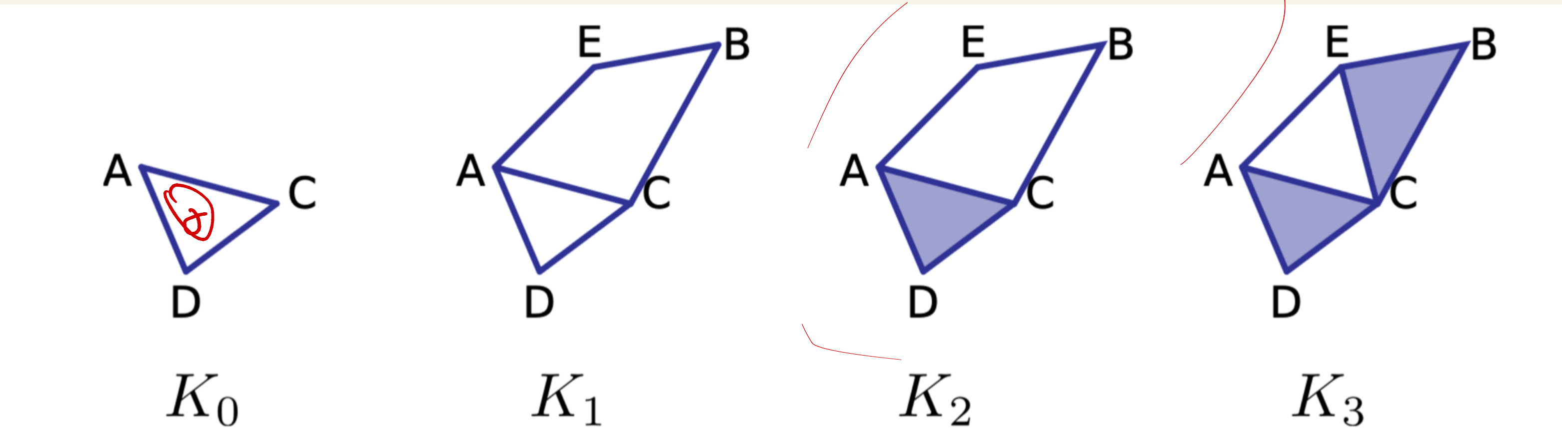


$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$$\begin{tikzcd}
{[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \\
 & {[B]} \arrow[ur] & & {[C]} \arrow[r] & {[C]} \arrow[ur] & {[D]} \arrow[r] & {[D]} \arrow[ur] & \\
 & & & & {[D]} \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{array}{cc} & {[A]} \\ \begin{matrix} {[A]} \\ {[B]} \end{matrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}$$

Another:



$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_2)$$

$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
{[ACD]} \arrow[r] & {[ACD]} \arrow[ur] & {[ABCE]} \arrow[r] & {[ACE]} \\
 & {[ABCE]} \arrow[ur] & &
\end{tikzcd}$$

The p^{th}-persistent homology groups are the images induced by inclusion:

$$H_p^{i,j} = \mathrm{Im}\left(H_p(K_i) \xrightarrow{f_p^{i,j}} H_p(K_j)\right)$$

$K_i \subseteq K_j$ $\quad$ $i \leq j$

The p^{th}-persistent Betti numbers are

$$\beta_p^{i,j} = \mathrm{rank}\left(H_p^{i,j}\right)$$

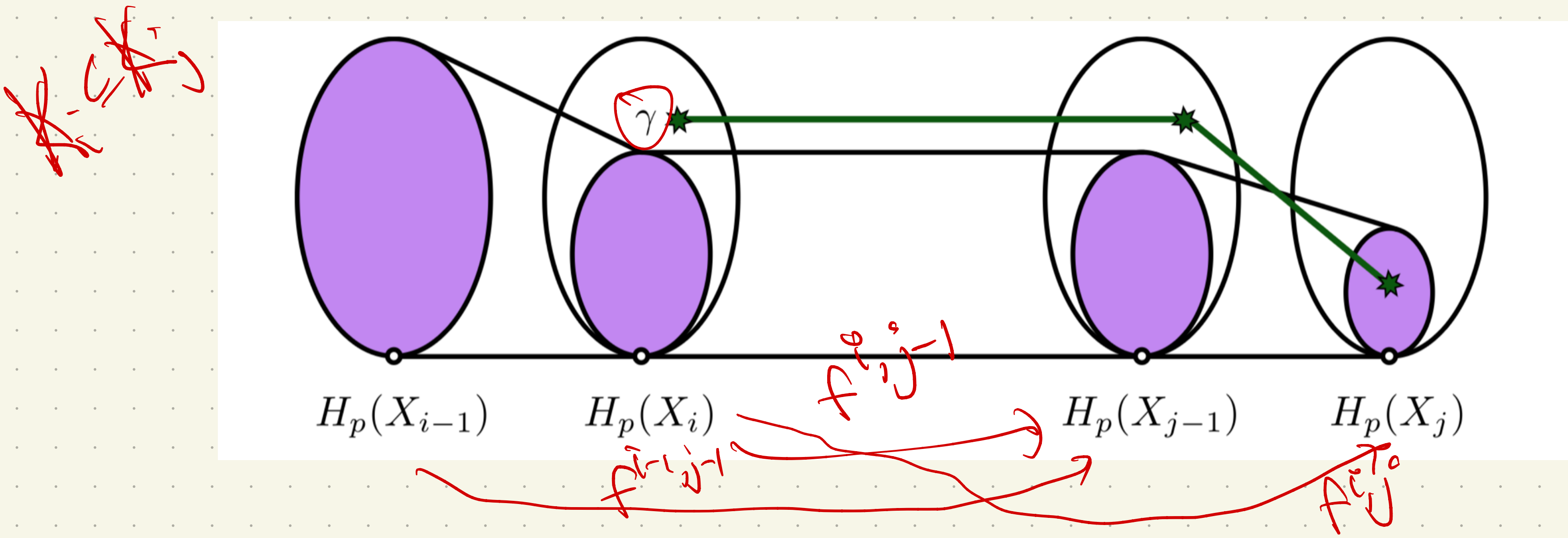
for a persistence module

$$H_p(K_0) \longrightarrow H_p(K_1) \longrightarrow \cdots H_p(K_i) \longrightarrow \cdots H_p(K_j) \to \cdots H_p(K_n)$$

Birth & death

We say a homology class $\gamma \in H_p(K_i)$ is born at K_i if it is not in $H_p^{i-1,i}$, & γ dies entering K_j if it merges with an older class, ie if $f_p^{i,j-1}(\gamma) \notin H_p^{i-1,j-1}$ but $f_p^{i,j}(\gamma) \in H_p^{i-1,j}$

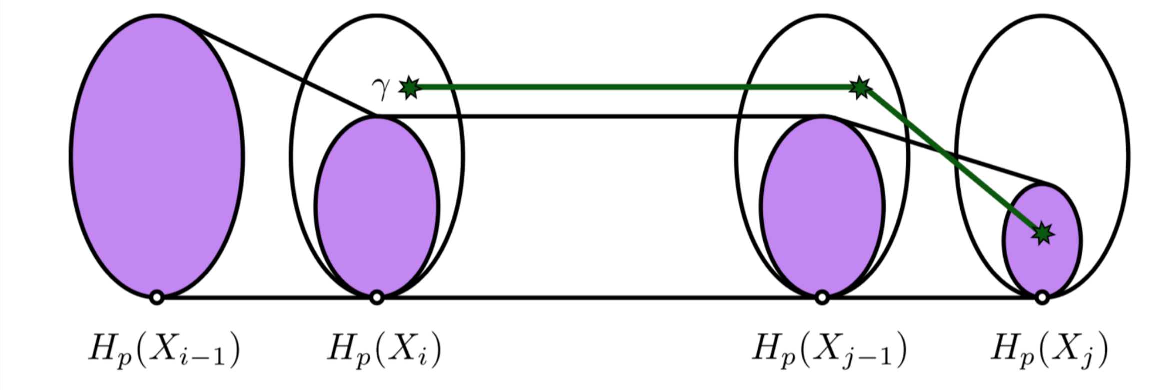
↑ Warning: not book's version!



Book's version of death:

γ dies entering K_j if

- $\gamma \in H_p(X_{j-1})$ is not trivial
- but $f\cancel{H}_p^{j-1,j}(\gamma) = 0$

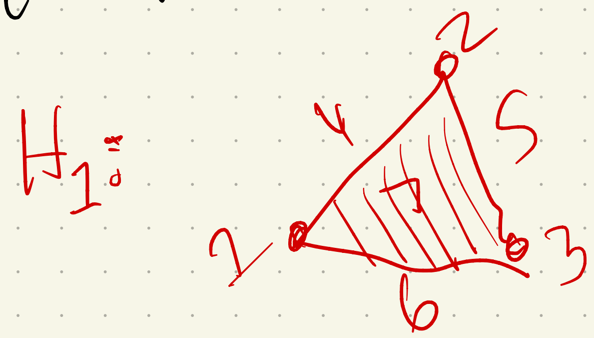


Only issue: no birth/death pairs in this definition

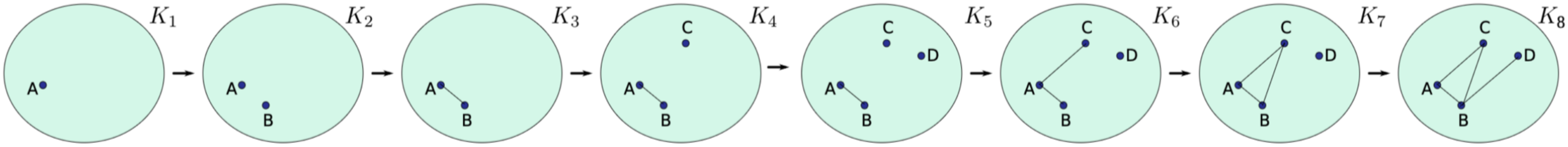
Pairing (book dfn)

Let $[c]$ be a p^{th} homology class that dies entering X_j. Then, it is born at X_i if + only if $\exists\ i_1 \leq i_2 \leq \cdots \leq i_k = i$ (with $k \geq 1$) s.t.

- $[c_{i_\ell}]$ is born at X_{i_ℓ} $(\ell \in [1..k])$
- $[c] = f_p^{i_1, j-1}([c_{i_1}]) + \cdots + f_p^{i_k, j-1}([c_{i_k}])$
- $i_k = i$ is smallest possible choice



Revisiting: when are births & deaths?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$$\begin{tikzcd}
{[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \\
 & {[B]} \arrow[ur] & & {[C]} \arrow[r] & {[C]} \arrow[ur] & {[D]} \arrow[r] & {[D]} \arrow[ur] & \\
 & & & & {[D]} \arrow[ur] & & &
\end{tikzcd}$$

b is born

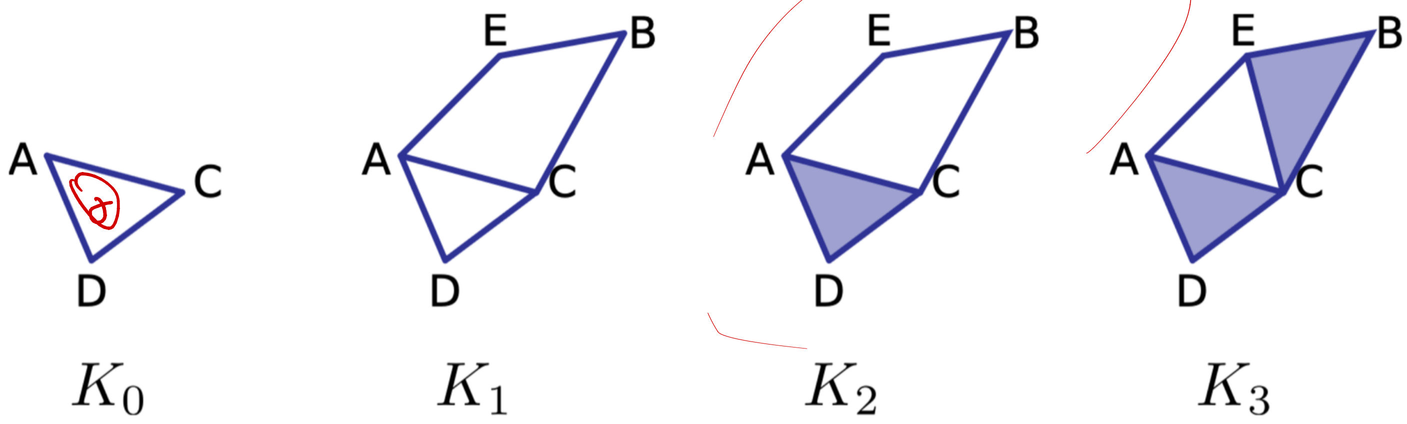
b dies

c is born

d is born

c dies

d dies



$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_2)$$

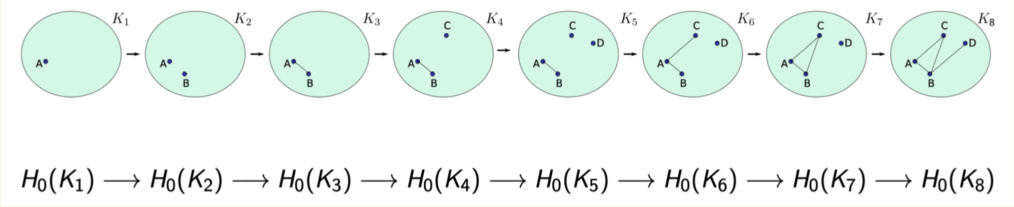
$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
{[ACD]} \arrow[r] & {[ACD]} \arrow[ur, bend left] & {[ABCE]} \arrow[r] & {[ACE]} \\
 & {[ABCE]} \arrow[ur, bend left] & &
\end{tikzcd}$$

Note: the maps $f_p^{i,j}$ change if basis changes or reorders

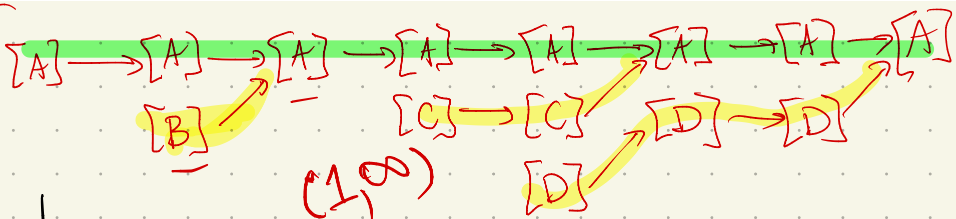
↳ but times are the same.

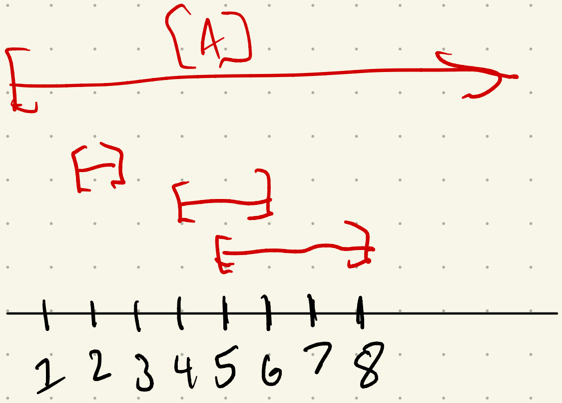
Result:

Given filtration:

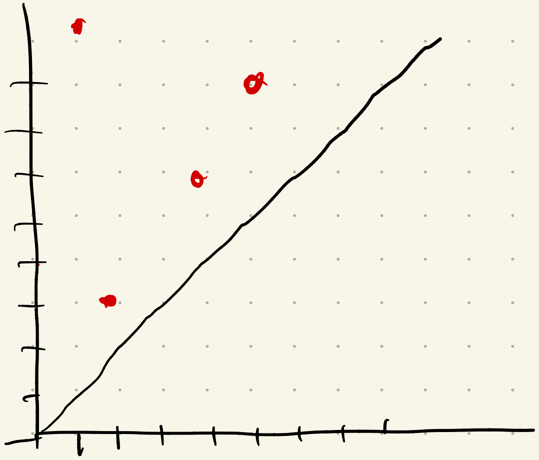


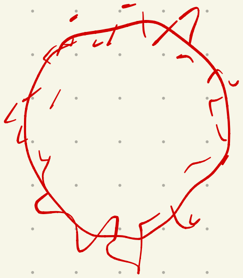
$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$





Barcodes





Persistence diagrams

More formally: Counting classes

$$\begin{tikzcd} 0 \arrow[r] & H_p(K_1) \arrow[r] & H_p(K_2) \arrow[r] \arrow[rr, bend left] & \cdots \arrow[r] & H_p(K_n) \arrow[r] & \overset{\infty}{\underset{K_{n+1}}{0}} \end{tikzcd}$$

- Attach 0 vector space at end
- Associate $n+1$ to $a_{n+1} = \infty$
- Then $\beta_p^{i,j}$ counts classes born before i which die after j are alive in j

How can we get # of classes born at i which die at j?

$$H_p^{i-1} \to H_p^{i} \to \cdots \to H_p^{j-1} \to H_p^{j}$$

Pairing function

for $0 < i < j \leq n+1$, define

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

→ # of classes born at i that die at j

Why?

$$\begin{tikzcd} H_p(X_{i-1}) \arrow[r, "f_p^{i-1,i}"] & H_p(X_i) \arrow[r, "f_p^{i,j-1}"] & H_p(X_{j-1}) \arrow[r, "f_p^{j-1,j}"] & H_p(X_j) \end{tikzcd}$$

When $\mu_p^{i,j} \neq 0$, the persitence of a class $[c]$, $Per([c])$, which is born at X_i + dies at X_j is defined as $a_j - a_i$.

↳ length of barcode

"lifetime"

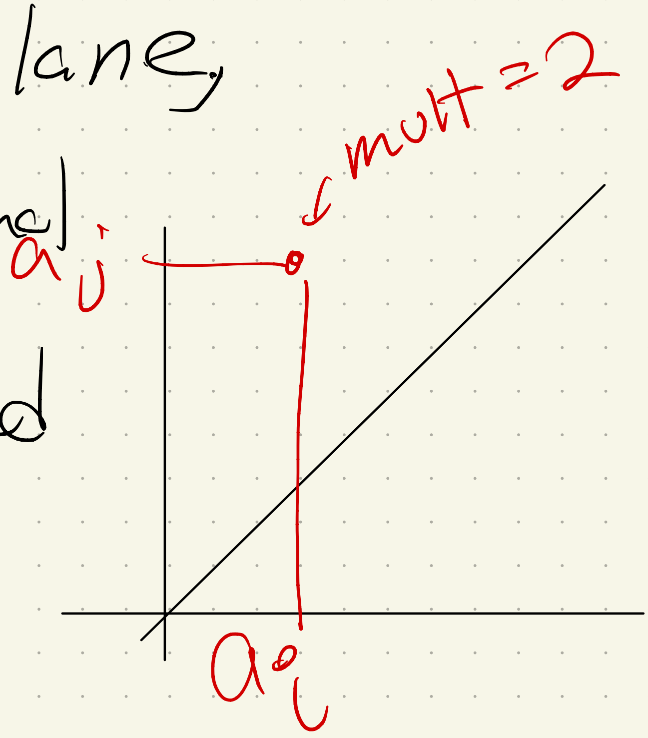
[If $j = n+1$ with $a_{n+1} = \infty$, $Pers([c]) = \infty$].

Persistence diagram $\mathrm{Dgm}_p(F)$

(also written $\mathrm{Dgm}_p(f)$)

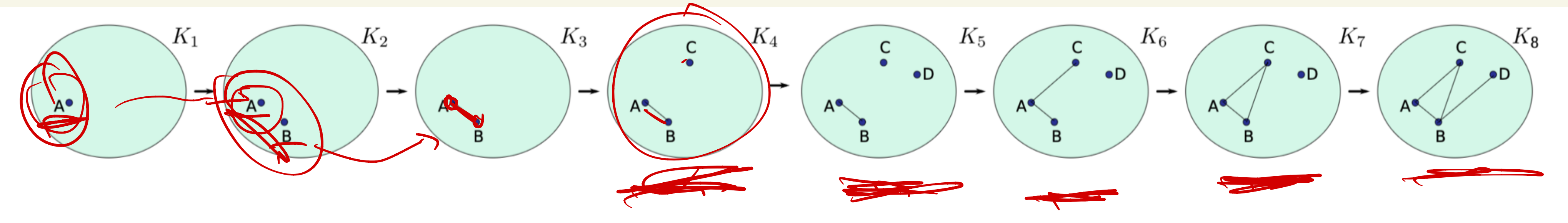
Filtration F on K induced by f.

$\mathrm{Dgm}_p(F)$ is obtained by drawing a point (a_i, a_j) with non-zero multiplicity $\mu_p^{i,j}$ $(i<j)$ on extended plane, where points on the diagonal $\Delta = \{(x,x) \in \mathbb{R}^2\}$ are added with infinite multiplicity



Let's try! First, calculate $\beta_0^{i,j}$

↳ then $\mu_0^{i,j}$



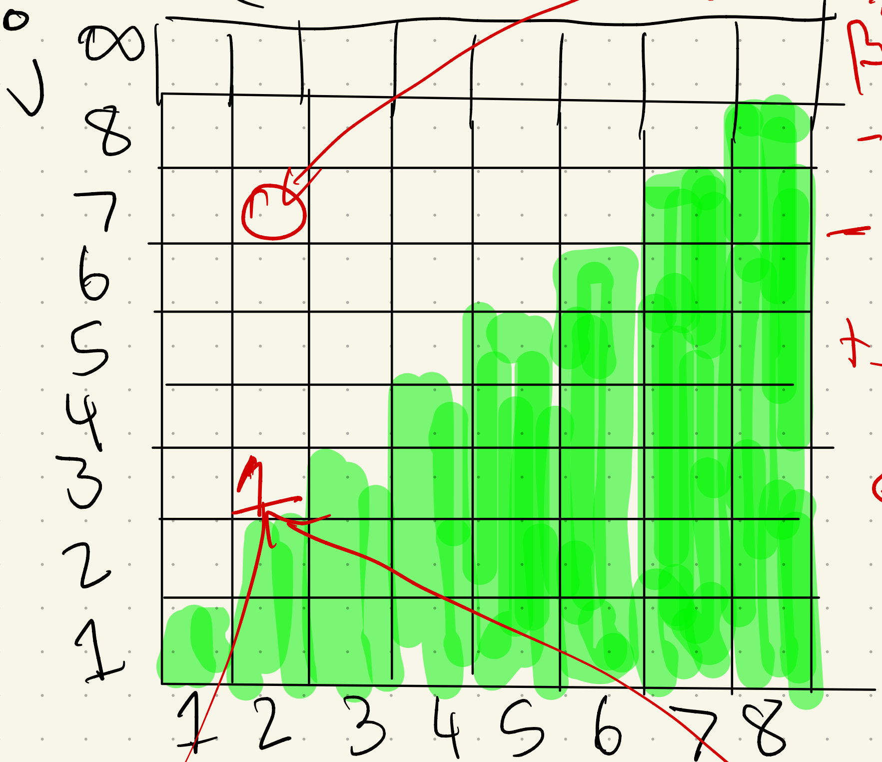
$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow \underline{H_0(K_3)} \longrightarrow H_0(K_4) \longrightarrow \underline{H_0(K_5)} \longrightarrow \underline{H_0(K_6)} \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$\beta_0^{i,j}$

$\overset{0}{j}$ ∞	0	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1
7	1	1	1	1	2	2	2	
6	1	1	1	1	2	2		
5	1	1	1	2	3			
4	1	1	1	2				
3	1	1	1					
2	1	2						
1	1							
	1	2	3	4	5	6	7	8 $\overset{0}{i}$

$$\mu^{i,j} = (\beta^{i,j-1} - \beta^{i,j}) - (\beta^{i-1,j-1} - \beta^{i-1,j})$$

j ∞	0	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1
7	1	1	1	1	2	2	2	
6	1	1	1	1	2	2		
5	1	1	1	2	3			
4	1	1	1	2				
3	1	1	1					
2	1	2						
1	1							
	1	2	3	4	5	6	7	8 i

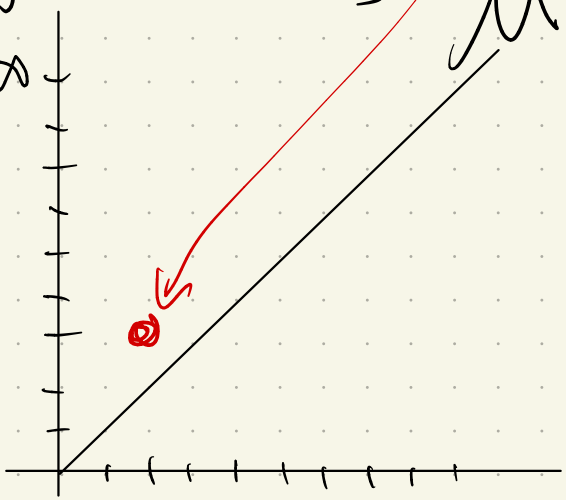


$\mu^{2,7} = \beta^{2,6} - \beta^{2,7} - \beta^{1,6} + \beta^{1,7}$

$\mu^{2,3} = \beta^{2,2} - \beta^{2,3} - \beta^{1,2} + \beta^{1,3}$

$\beta^{i,j}_\infty \longrightarrow \mu^{i,j} \quad (i<j)$

Then Dgm(F) =



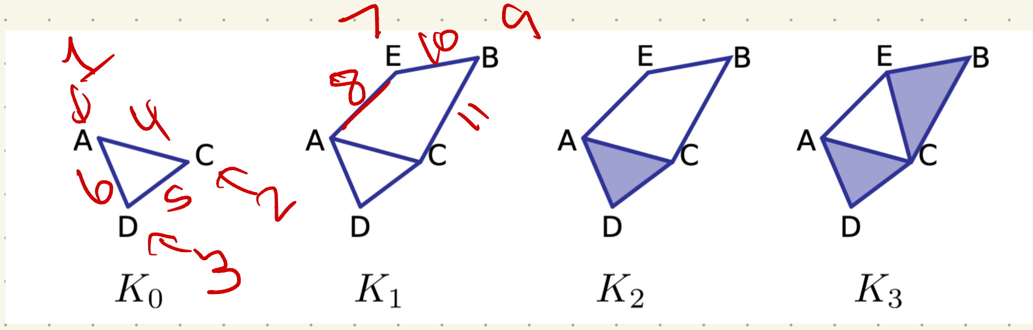
OK, let's avoid ever doing this by hand again...

Let $f: K \to \mathbb{N}$ give the index where a simplex σ appears in filtration.

A compatible ordering of the simplices is a sequence $\sigma_1, \sigma_2, \dots, \sigma_m$ s.t.

- $f(\sigma_i) < f(\sigma_j) \Rightarrow i<j$
- $\sigma_i \leq \sigma_i \Rightarrow i<j$

Ex:



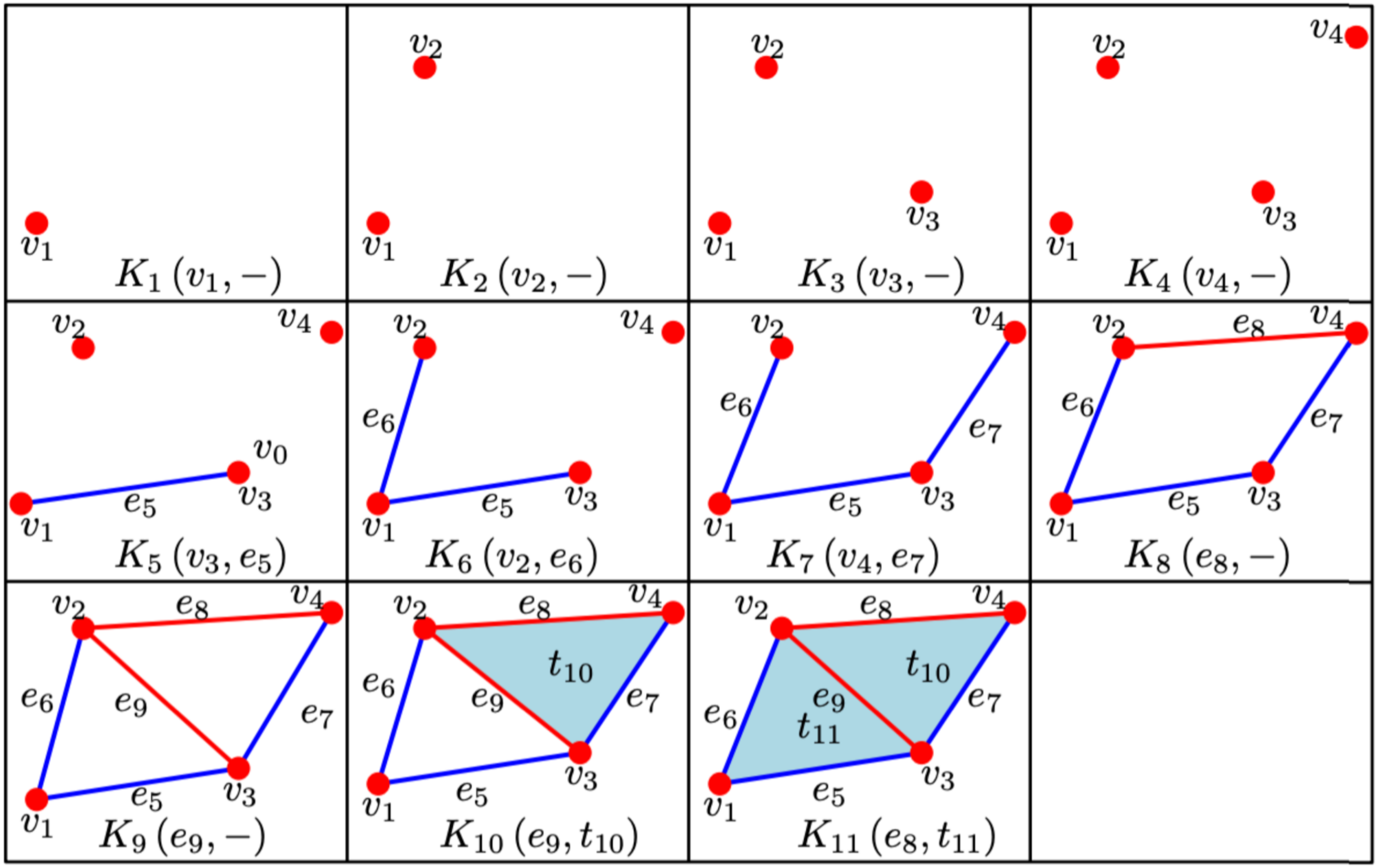
Essentially, we now have a simplex-wise filtration: assume $K_j / K_{j-1} = \sigma_j$ is a single simplex.

When p-simplex σ_j is added, two possibilities:

① A non-boundary p-cycle c along with its classes $[c]+h$ for $h \in H_p(K_{j-1})$ are born. Call σ_j positive (or a creator).

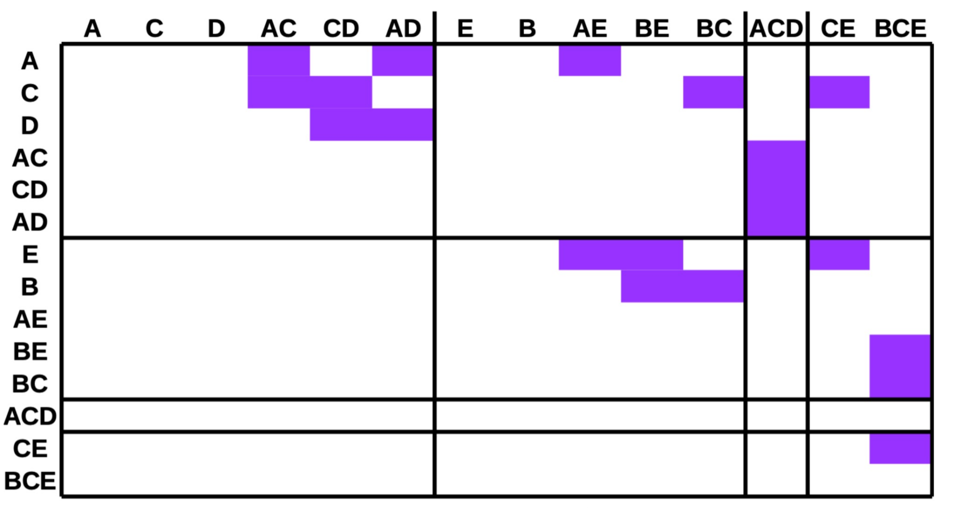
② An existing (p-1)-cycle c along with its class $[c]$ dies. Call σ_j negative (or a destroyer).

Examples


v_1
$K_1\,(v_1,-)$
v_2
v_1
$K_2\,(v_2,-)$
v_2
v_3
v_1
$K_3\,(v_3,-)$
v_2
v_4
v_3
v_1
$K_4\,(v_4,-)$
v_2
v_4
v_0
v_3
e_5
v_1
$K_5\,(v_3,e_5)$
v_2
v_4
e_6
e_5
v_3
v_1
$K_6\,(v_2,e_6)$
v_2
v_4
e_6
e_7
e_5
v_3
v_1
$K_7\,(v_4,e_7)$
v_2
e_8
v_4
e_6
e_7
e_5
v_3
v_1
$K_8\,(e_8,-)$
v_2
e_8
v_4
e_6
e_9
e_7
e_5
v_3
v_1
$K_9\,(e_9,-)$
v_2
e_8
v_4
t_{10}
e_6
e_9
e_7
v_3
e_5
v_1
$K_{10}\,(e_9,t_{10})$
v_2
e_8
v_4
t_{10}
e_9
t_{11}
e_6
e_7
v_3
e_5
v_1
$K_{11}\,(e_8,t_{11})$

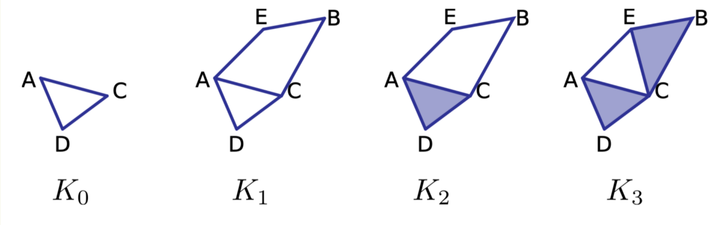
An algorithm

Take boundary matrix, with rows & columns in simplex-wise order:



	A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A				■		■			■					
C				■	■						■		■	
D					■	■								
AC												■		
CD												■		
AD												■		
E									■	■			■	
B										■	■			
AE														
BE														■
BC														■
ACD														
CE														■
BCE														

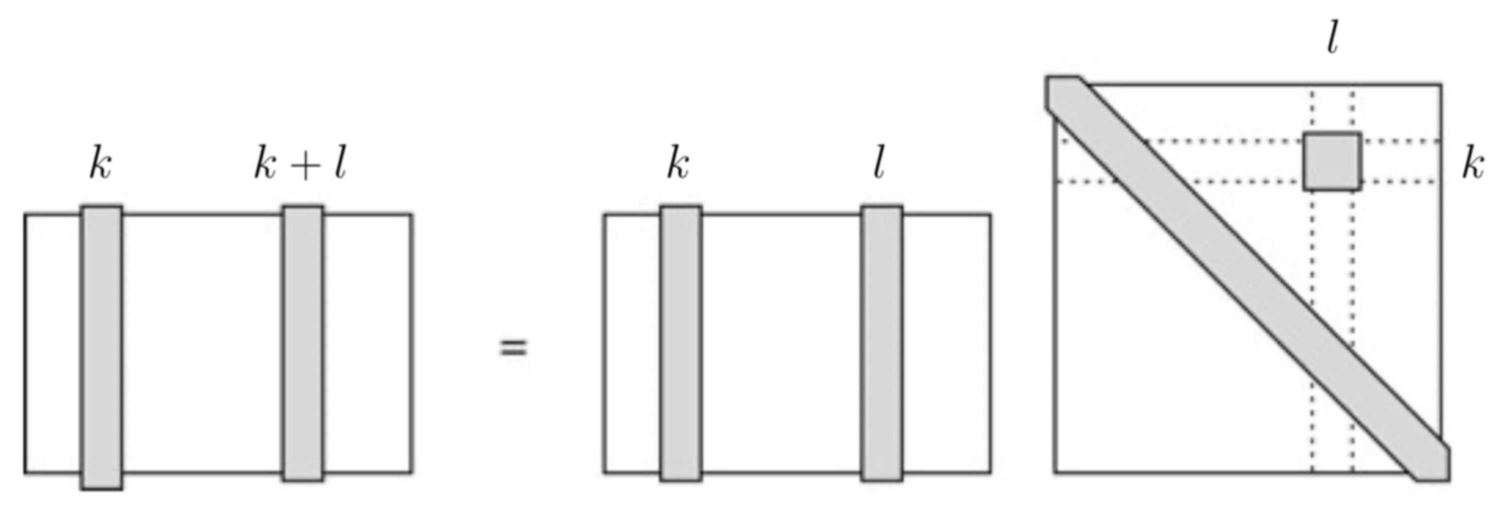
K_0 K_1 K_2 K_3



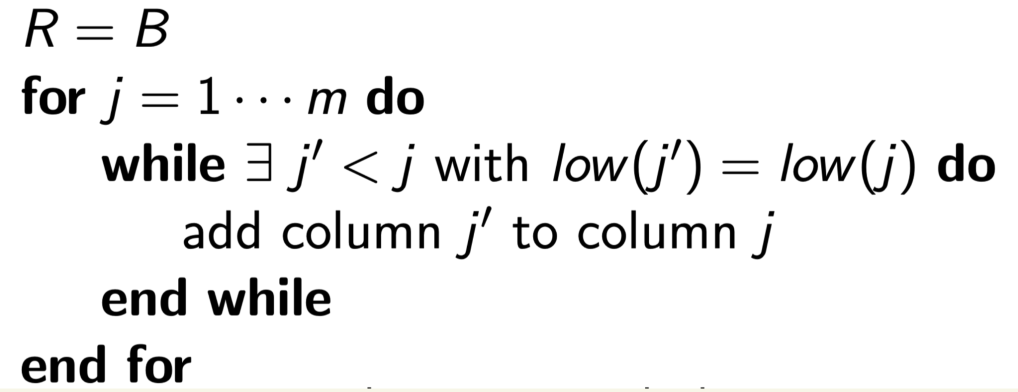
- Let low(j) = row of lowest 1 in column j (& if all 0's, low(j) = NaN)
- R is reduced if low(j) ≠ low(j') for any j ≠ j

Matrix operations

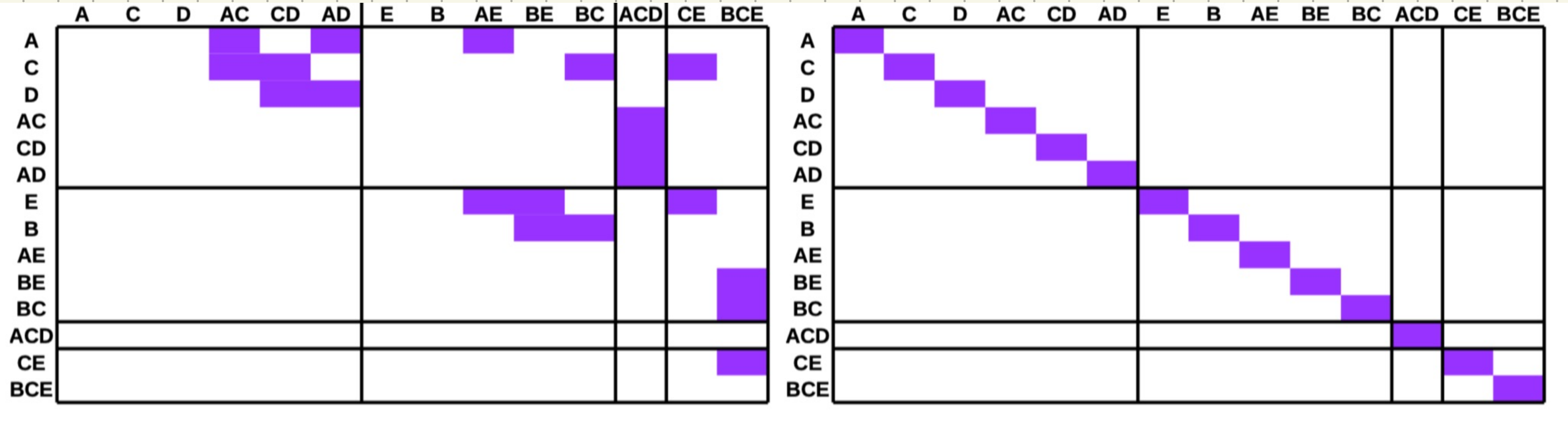
To add row k to row l, can create matrix with 1 in l, k:



Here!



```
$R = B$
for $j = 1 \cdots m$ do
    while $\exists\ j' < j$ with $low(j') = low(j)$ do
        add column $j'$ to column $j$
    end while
end for
```



	A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A				■		■			■					
C				■	■						■		■	
D					■	■								
AC												■		
CD												■		
AD												■		
E									■	■			■	
B										■	■			
AE														
BE														■
BC														■
ACD														
CE														■
BCE														

	A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A	■													
C		■												
D			■											
AC				■										
CD					■									
AD						■								
E							■							
B								■						
AE									■					
BE										■				
BC											■			
ACD												■		
CE													■	
BCE														■

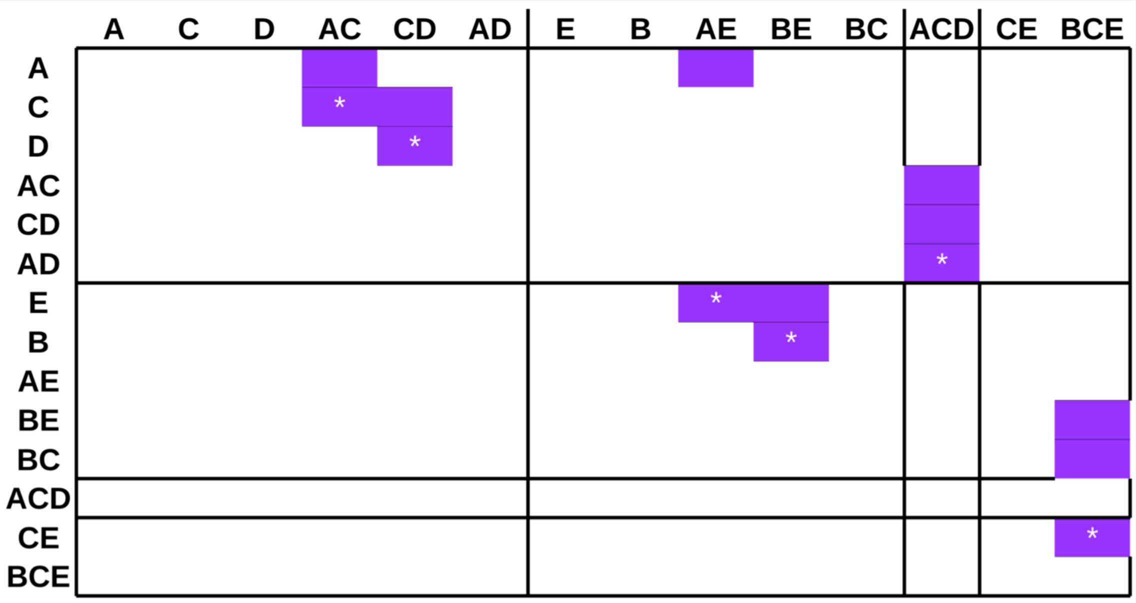
Idea

- B is upper triangular, + if we add from left it stays that way
- If a column is entirely 0, that simplex created a homology class (so it is positive)
- If a column has a lowest 1, then this simplex killed a class from the previous step.

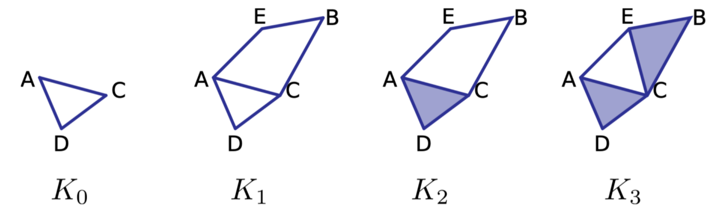
Pairing

- Every negative simplex must be paired with a previous positive (birth/death)

↳ pair with its lowest 1



	A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A				■					■					
C				■ *	■									
D					■ *									
AC												■		
CD												■		
AD												■ *		
E									■ *	■				
B										■ *				
AE														
BE														■
BC														■
ACD														
CE														■ *
BCE														

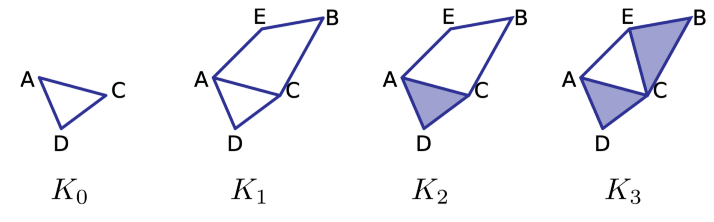


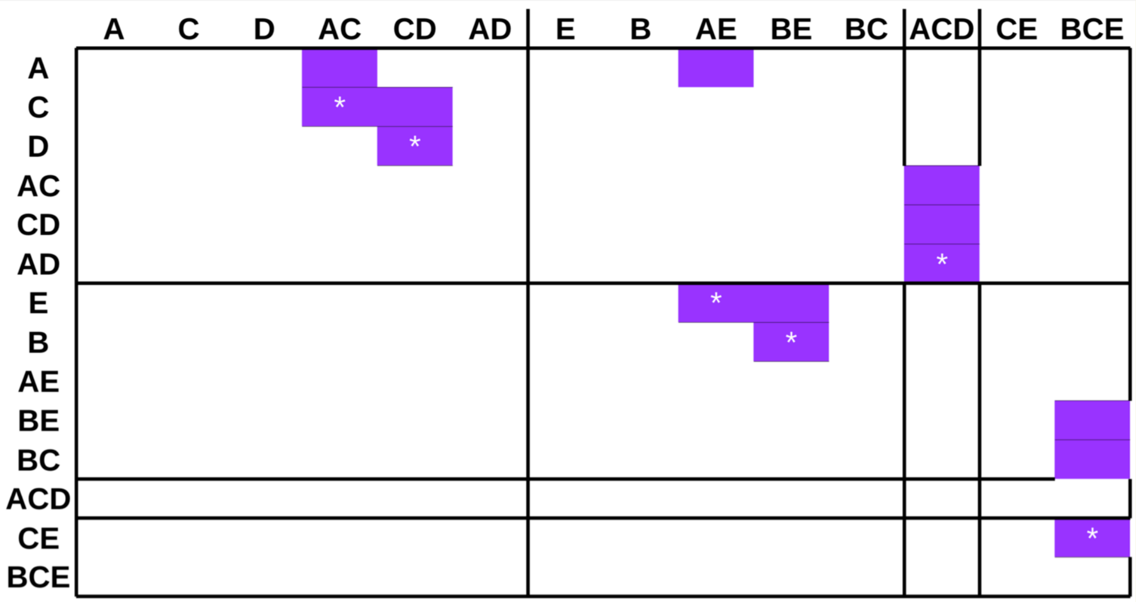
Pairs:

Fact

The number of unpaired p-simplices in a simplex-wise filtration of K is its pth Betti number.

So: use pairs to build persistence diagram.





	A	C	D	AC	CD	AD	E	B	AE	BE	BC	ACD	CE	BCE
A				▨					▨					
C				▨ *	▨									
D					▨ *									
AC												▨		
CD												▨		
AD												▨ *		
E									▨ *	▨				
B										▨ *				
AE														
BE														▨
BC														▨
ACD														
CE														▨ *
BCE														

→

History

Matrix algorithm is from

Edelsbrunner-Letscher-Zomorodian 2000

Algebraic formulation given in

Carlsson & Zomorodian 2004

Independent formulations

Frosni 1990 - manifold comparison in Euclidean space

Robbins 1999 - crystaline structures & periodicity