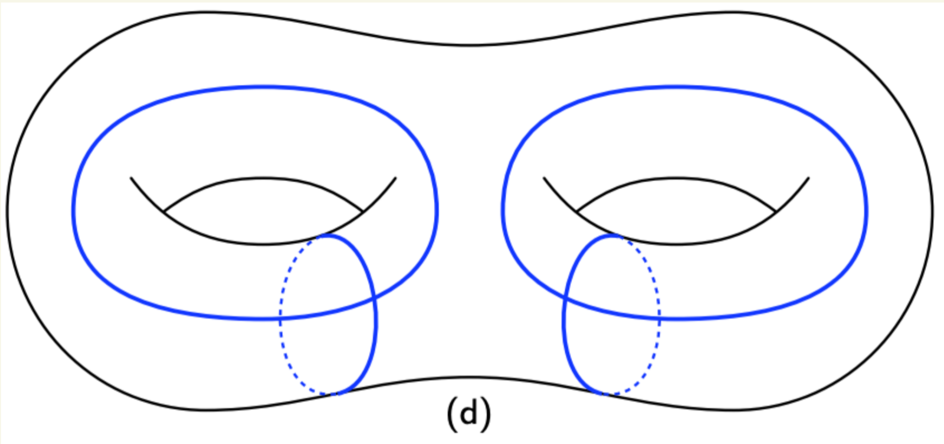
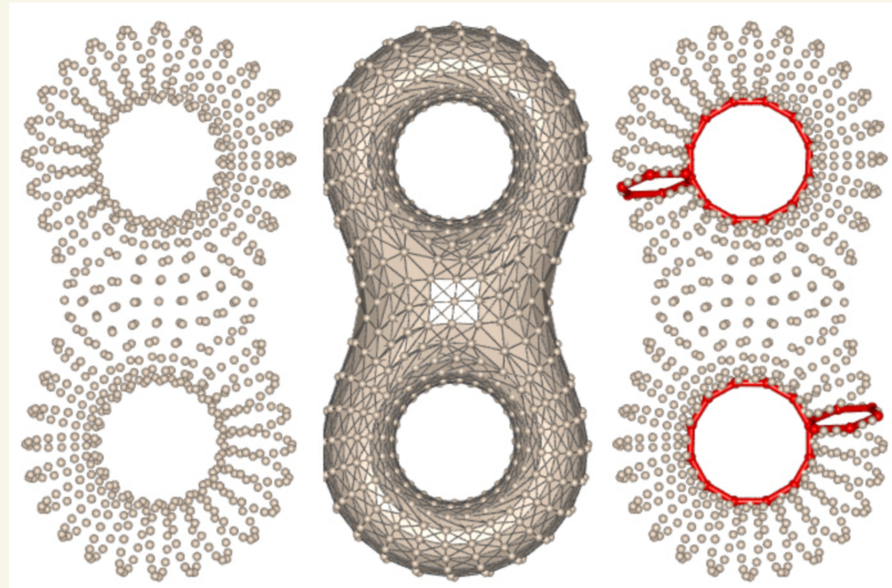
TDA - fall 2025

Optimal Cycles (cont)

Last time: Optimal homology basis



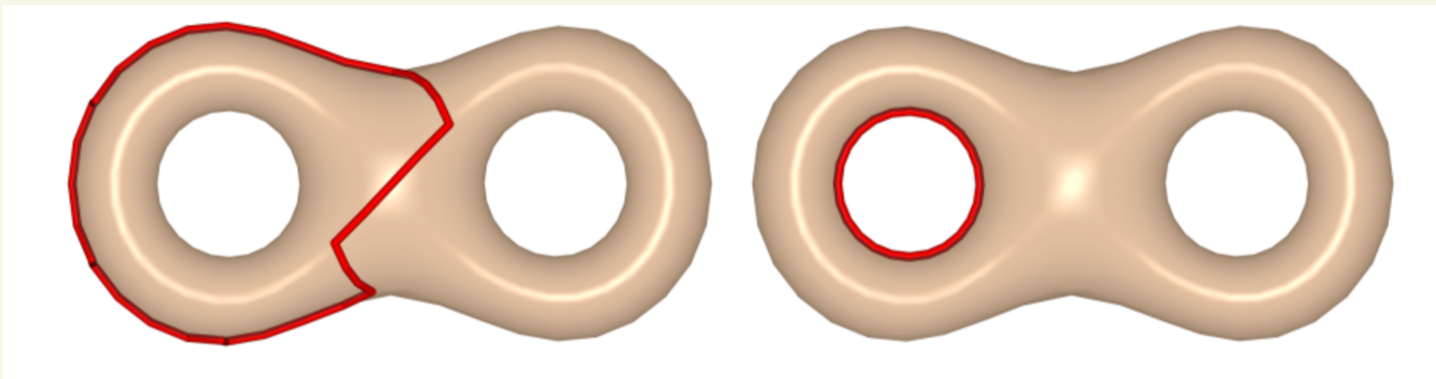


Given a weight function, find the set of cycles with min total weight which generate H_p.

In general:

Related question: homology localization

Given a p-cycle c, find minimum weight cycle c' such that $[c'] = [c]$.



Interestingly:

- For $\mathbb{Z}_2$-homology, NP-Hard
- With $\mathbb{Z}$-coefficients, polynomial time (if no torsion)

Algorithm ($\mathbb{Z}$-homology)

Reduce to integer programming

Given p-chain $x = \sum_{i=0}^{m-1} x_i \sigma_i$, $x_i \in \mathbb{Z}$,

let $\vec{x} \in \mathbb{Z}^m$ be $\vec{x} = (x_0, \dots, x_i, \dots, x_{m-1})$

Recall: $\|x\|_1 = \sum_{i=0}^{m} |x_i|$

& D_p be boundary matrix $\partial_p : C_p \to C_{p-1}$

Let W be weight matrix:

$$\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Why?

Take cycle $\vec{x}$.

$$W\vec{x} = \begin{bmatrix} w_{\sigma_1} & & & \\ & w_{\sigma_2} & & \\ & & \ddots & \\ & & & w_{\sigma_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Then $|W\vec{x}| =$

Then: ILP is

Given a p-chain c, weights W,

$$\underset{x,y}{\text{minimize}} \quad \|Wx\|_1$$

$$\text{s.t.} \quad x = c + D_{p+1} y$$

$$x \in \mathbb{Z}^m$$

$$y \in \mathbb{Z}^n$$

wher m = # of p-simplices

& n = # of $(p+1)$-simplices

Problem: $\underbrace{\text{Integer}}$ Linear Programming

But:

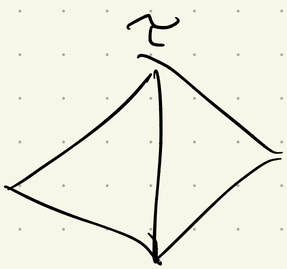
If determinant of every square submatrix is $0, \pm 1$, then matrix is totally unimodular

Fact: If a matrix is totally unimodular, then the LP also solves the ILP.

$\Rightarrow$

Claim: D_{p+1} is totally unimodular when K triangulates a $(p+1)$-dim compact orientable manifold

Why? • Each p-simplex is facet of ≤ 2 $p+1$ simplices



$\Rightarrow$ each row τ

• Known sufficiency conditions for 0-1 matrices work for D_{p+1}

$\boxed{\text{Heller-Tompkins 56}}$

Torsion

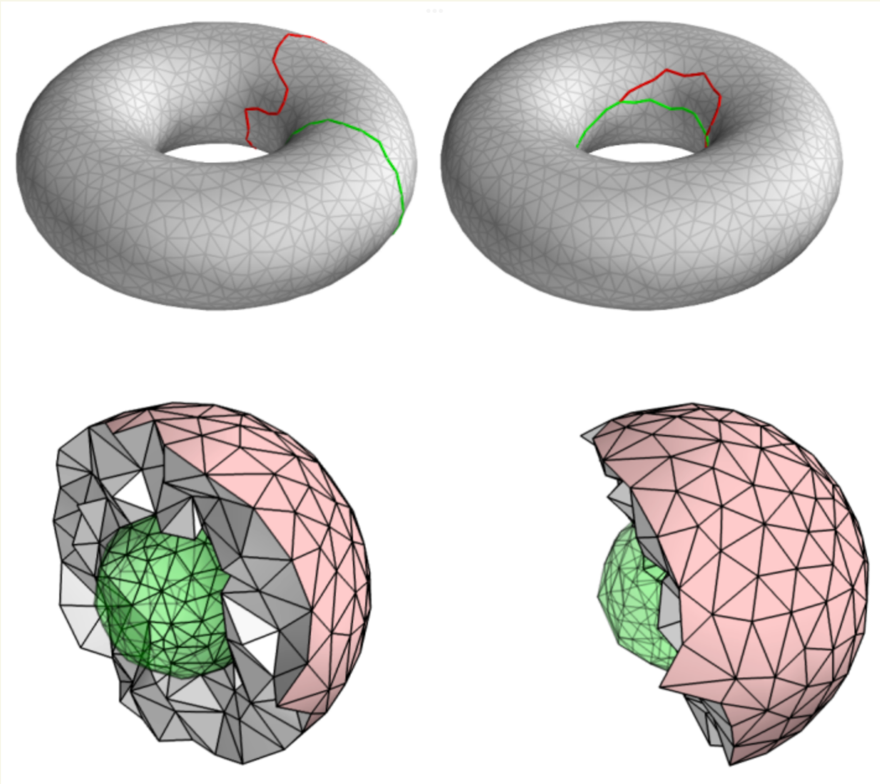
Unfortunately, not 0,1-matrix for D_i with $i < p+1$, + fails for $\mathbb{Z}_2$ homology entirely.

More generally: Any group G can be written as $G = F \oplus T$

- $F \cong (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})$
- $T \cong (\mathbb{Z}/t_1 \oplus \cdots \oplus \mathbb{Z}/t_k)$

torsion subgroup

Theorem: D_{p+1} is totally unimodular
$\iff H_p(L, L_0)$ is torsion-free
for all pure subcomplexes L_0 & L
in K of dimensions p & $p+1$ respectively,
where $L_0 \subset L$

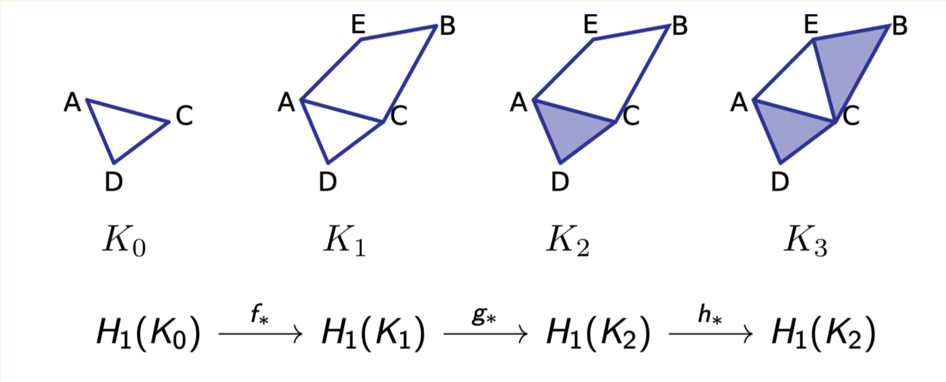


Dey-Hirani-Krishnamoorthy 2011

Optimal persistent cycles

Inside a filtration, how to get "best" cycle in a persistent homology class?

Recall: had barcodes or diagrams but many choices of representative:



Again, assume simplices have a weight function $w: K^p \to \mathbb{R}_{\geq 0}$ (K^p the p-simplices).

We say a cycle $c = \sum_{\sigma_i \in K_p} \alpha_i \sigma_i$, $\alpha_i \in \mathbb{Z}_2$, is a persistent cycle for $[b, d)$ if c is born at $K_{\leq b}$ and becomes a boundary in K_d.

A cycle is optimal if it has the least weight for all such cycle for a bar $[b, d)$.

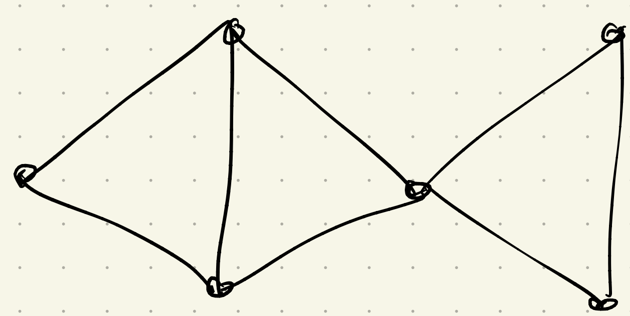
[Note: could have $d = \infty$, then no death]

Unfortunately, NP-Hard in general (even just H_1). Dey-Hou-Mandal 2018

However, possible in some cases:

A simplicial complex K is a weak $(p+1)$-pseudomanifold if each p-simplex is a face of no more than 2 $(p+1)$-simplices.

Example:



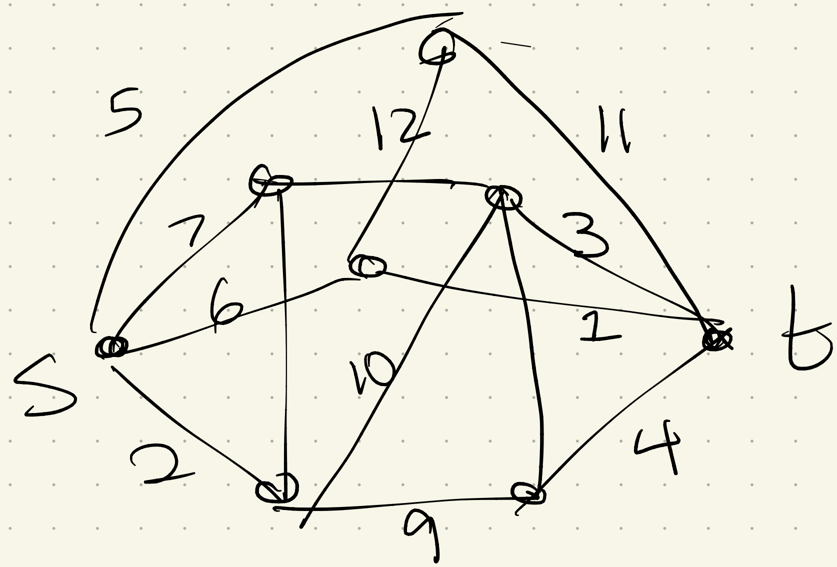
Algorithm for finite intervals & weak (pH)-pseudomanifolds:

Based on cuts:

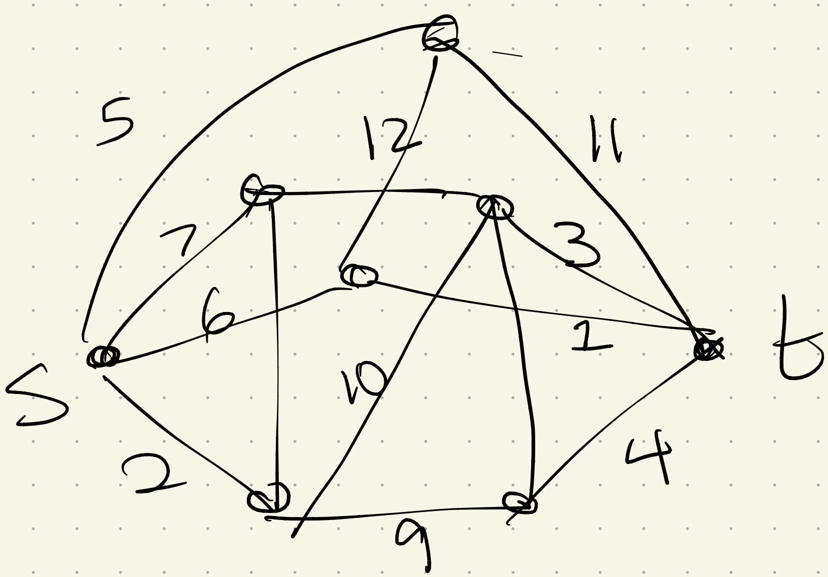
Given $G=(V,E)$ with 2 designated vertices s & t, & a capacity $c(e) \in \mathbb{R}_{\geq 0}$.
(or can have a collection of source vertices & sink vertices)

A cut is a partition of V into (S,T) s.t.

- $S \cup T = V$
- $s \in S$, $t \in T$

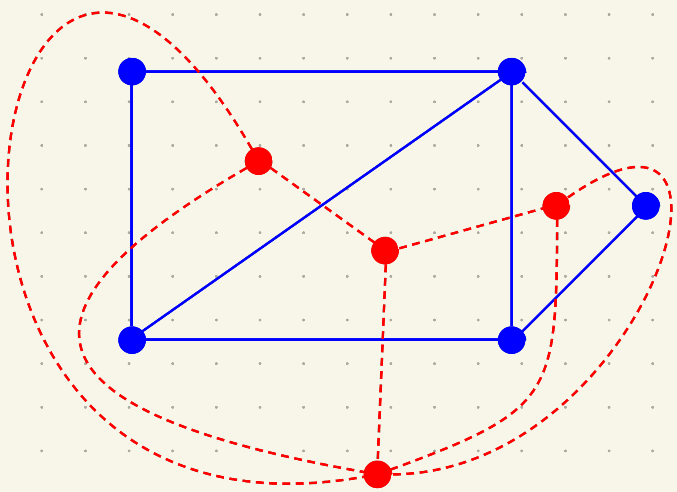


The capacity of a cut (S,T) is defined as $\sum_{\substack{u \in S \\ v \in T \\ uv \in E}} c(uv)$



Dual graphs

In a planar graph, the dual graph is a well studied object:



$G = (V, E)$ (in blue)
with faces

G^*:

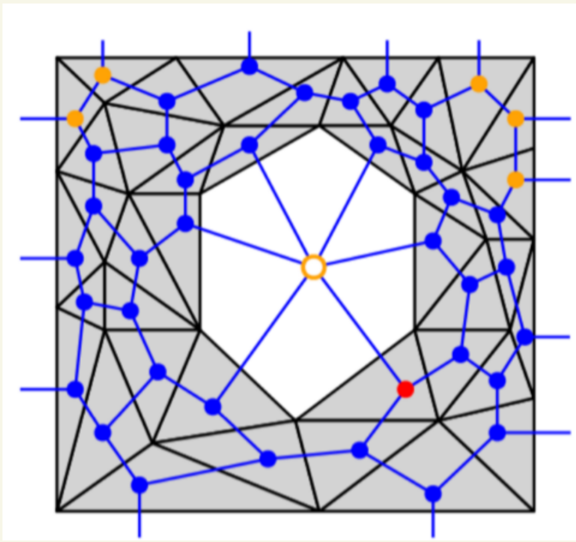
From persistent cycles to cuts

Build a dual complex!

sinks

source

dual complex



vertices: $(p+1)$-simplices

edges: p-simplices

plus infinite vertex for boundary $(p+1)$-simplices

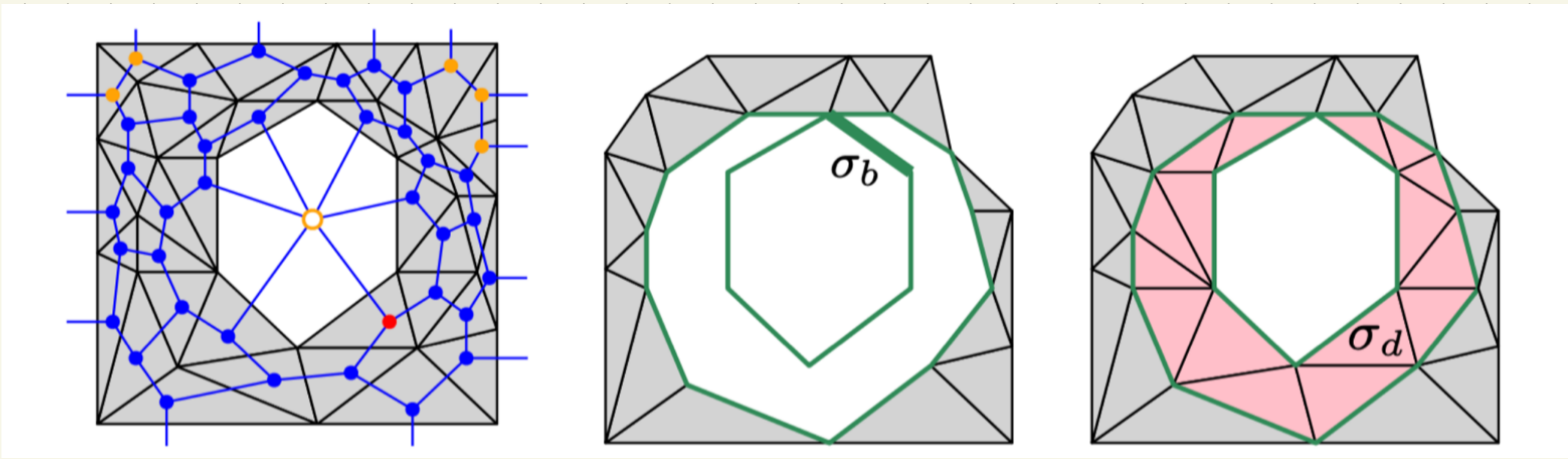
If σ_b & σ_d are creator & destructor simplices, make σ_d the source & sink in V_∞ plus simplices not in K_d.

Edge capacities:

- if σ_b or before, capacity = weight.
- otherwise $= \infty$

So, what is a cut here?

Intuition



Consider a persistent $[b,d)$ cycle c

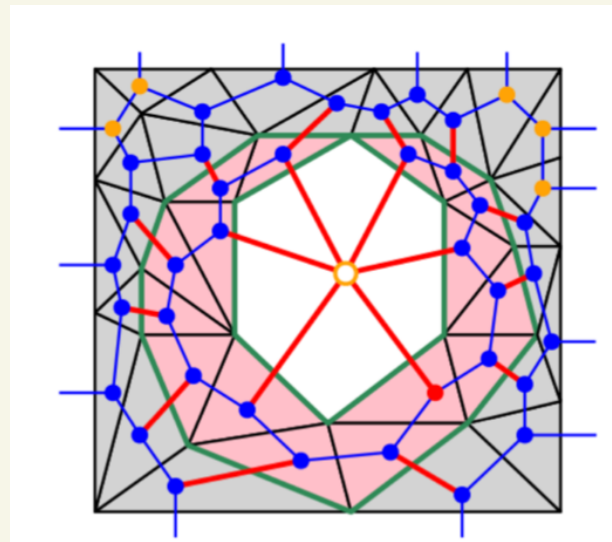
$\Rightarrow \exists$ $(p+1)$-chain A in $K_{\leq d}$ created by σ_d s.t. $\partial A = c$

So: let $S =$

$T =$

Then:

- (S,T) has finite capacity: which edges cross it?



In fact: any (S,T) cut must yield a persistent cycle in $[b,d)$, since can't use the ∞-edges

Result Works well on many data sets:

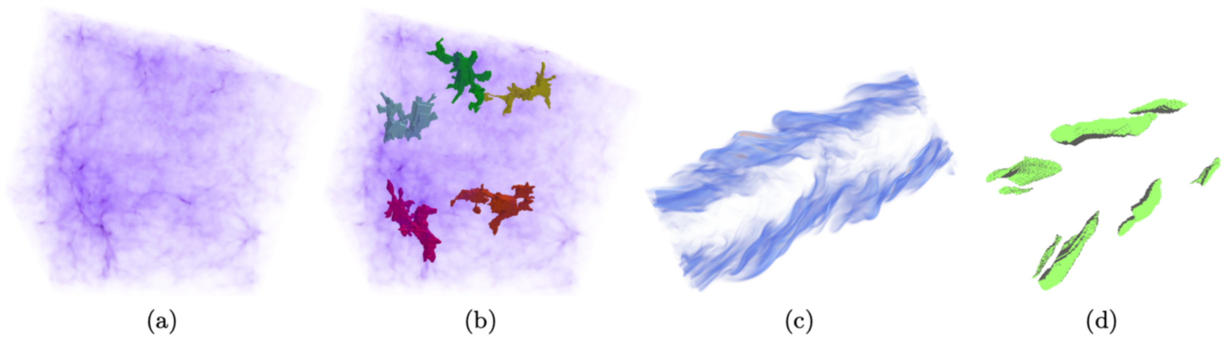


Figure 4: (a,b) Cosmology dataset and the minimal persistent 2-cycles of the top five longest intervals. (c,d) Turbulent combustion dataset and its corresponding minimal persistent 2-cycles.

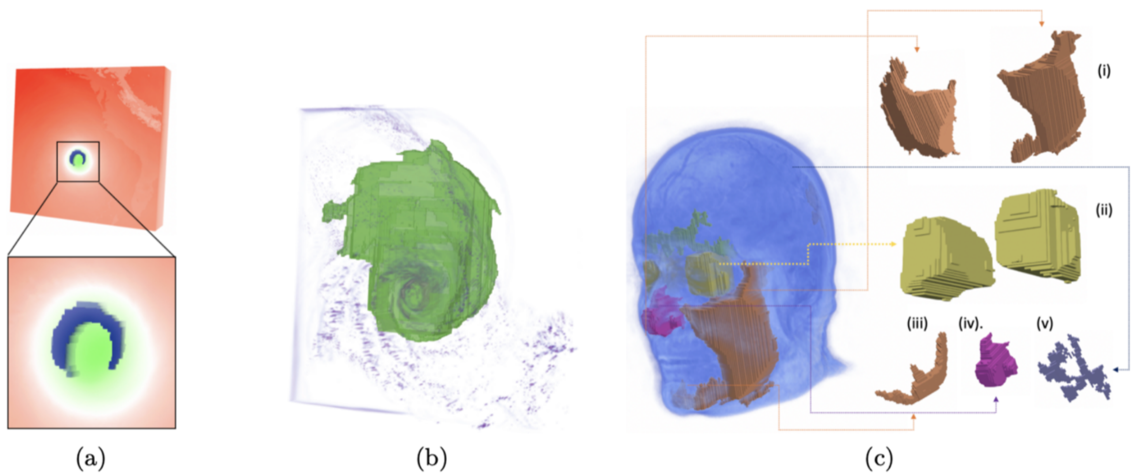


Figure 5: (a,b) Minimal persistent 2-cycles for the hurricane model. (c) Minimal persistent 2-cycles of the larger intervals for the human skull. i: Right and left cheek muscles with the right one rotated for better visibility. ii: Right and left eyes. iii: Jawbone. iv: Nose cartilage. v: Nerves in the parietal lobe.

Takeaway

Can connect some interesting classical graph algorithms to homology!