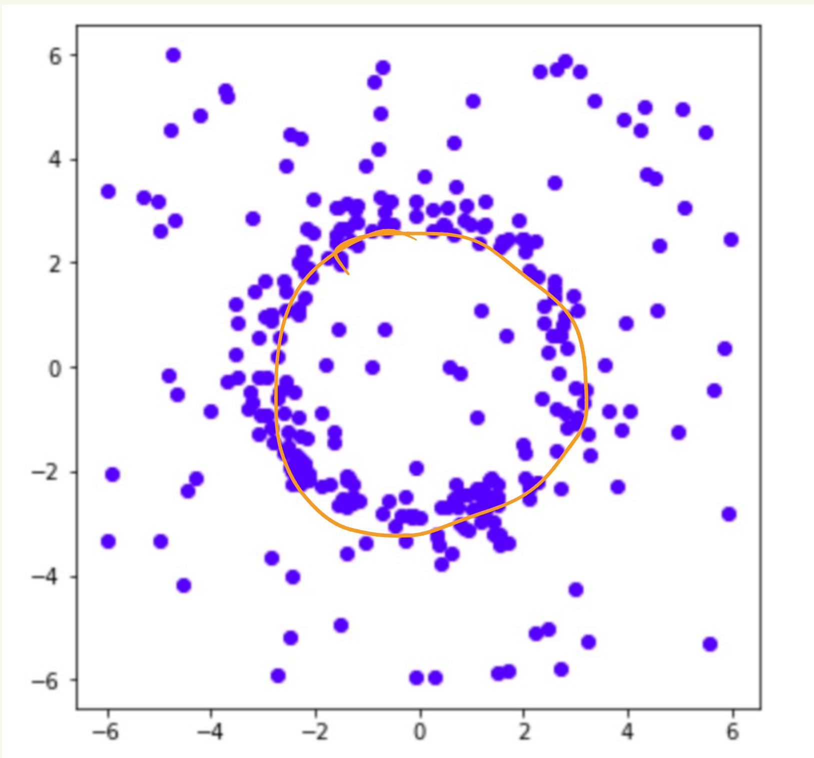
TDA - fall 2025

Multiparameter
Persistence

1-parameter persistence:

One problem: outliers

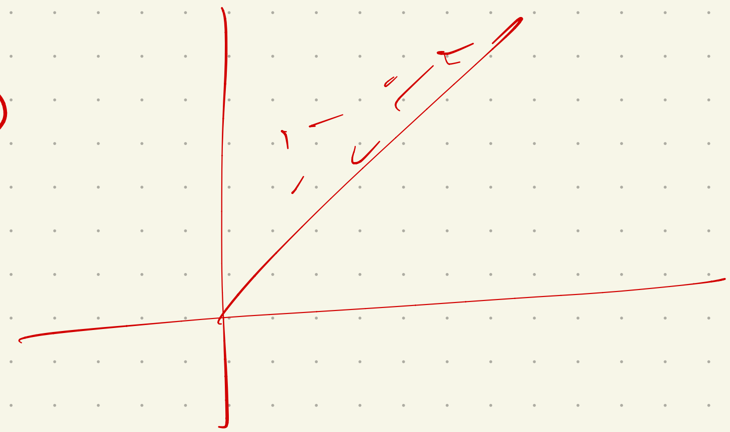
What will PD be?



want:

x

PD

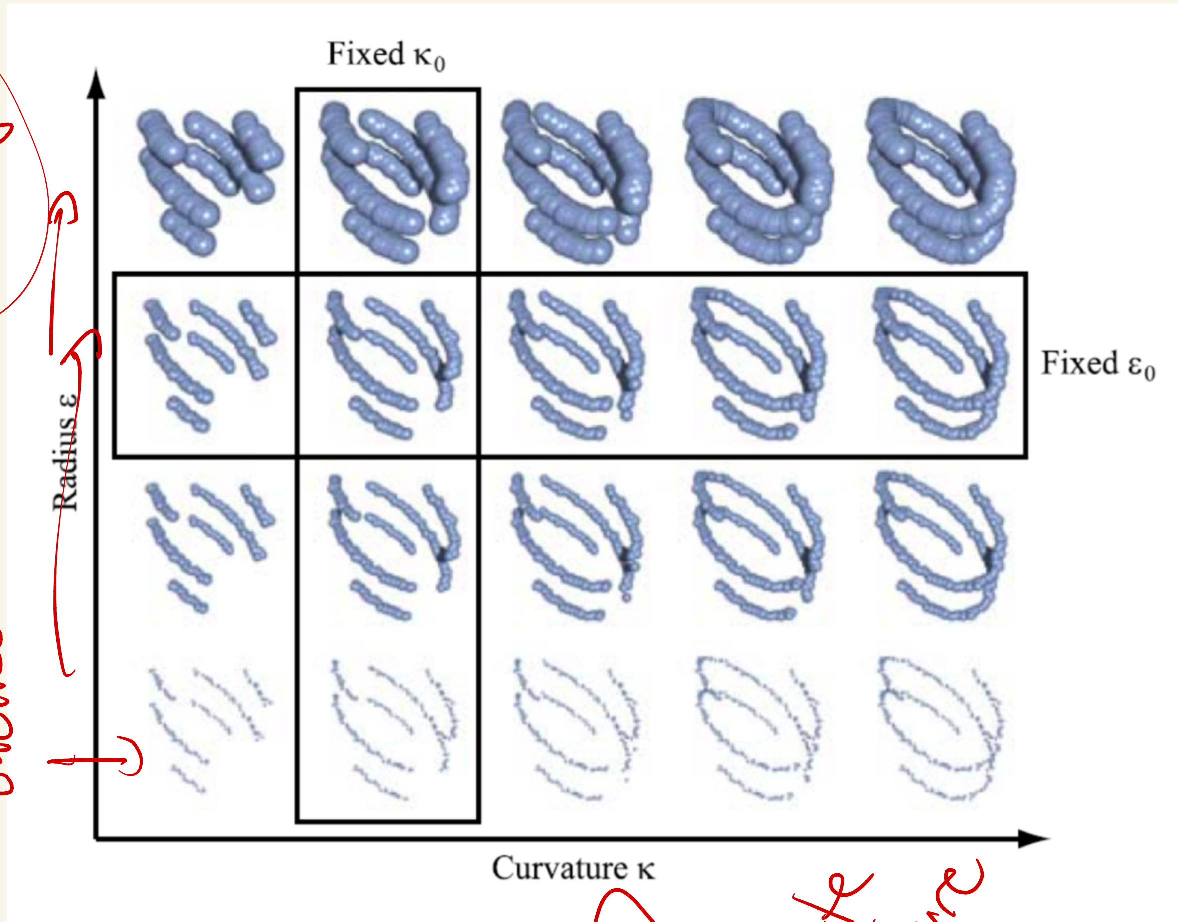


Motivation:

More principled way to bring in density to filtration?

Bifiltrations

2 different filtrations



Euclidean data

discrete curvature

Common example

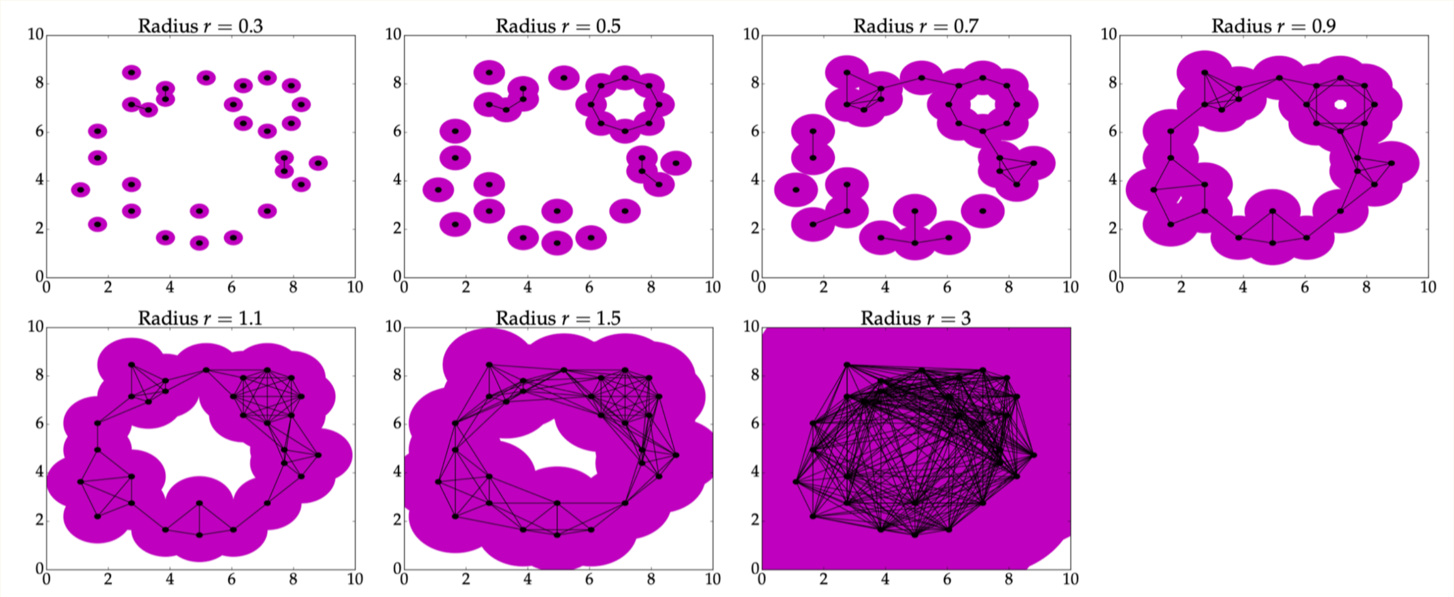
P = finite metric space

$N(P)_r$ = r-neighborhood graph of P:

- Vertex set P, and edge $(i,j) \in N(P)_r$ iff $d(i,j) \leq r$
- if $r < 0$, $N(P) = \emptyset$

$R(P)_r$:

Vietoris-Rips complex



$\longrightarrow$

Then, add some other filtration:

- Curvature estimate K

- Density: $\gamma_r(x) = C \cdot$ (# pts within distance r of x)

  where C is normalization constant so $\sum \gamma(x) = 1$

- Gaussian density: $\sigma > 0$ parameter, C again normalization constant:

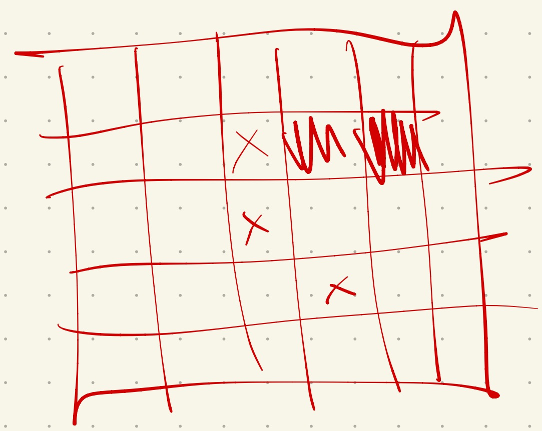
$$\gamma(x) = C \sum_{y \in P} \exp\left(-\frac{d(x,y)^2}{2\sigma}\right)$$

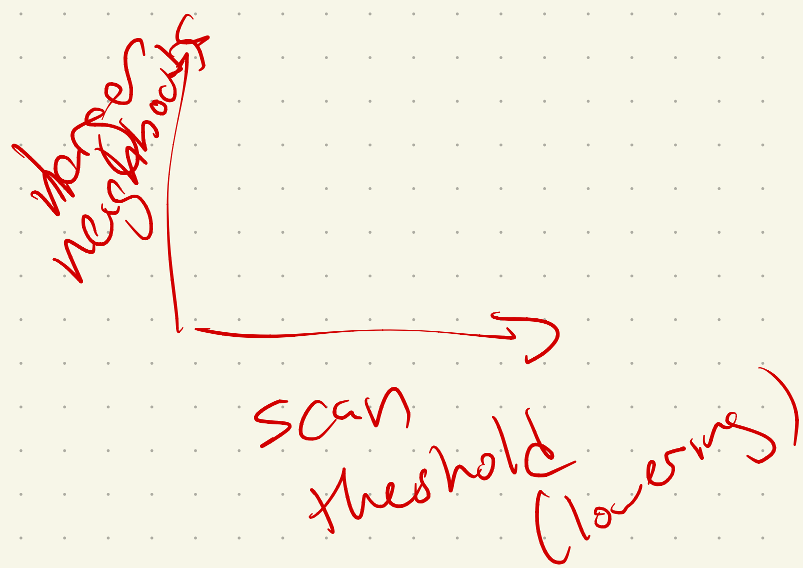
Or, on voxelized data:

often have intensity

↳ filter based on voxels with $\leq t$ intensity

↳ or, fixing t, can filter based on distance to nearest voxel.





More formally:

A **bifiltration** is a collection of simplicial complexes indexed by $\mathbb{Z}^2$ ← or $\mathbb{R}^2$

s.t. $F_a \subseteq F_b$ when $a \leq b$ $\quad a,b \in \mathbb{Z}^2$

Here, $(a_1, a_2) \leq (b_1, b_2)$ if

$$
\begin{tikzcd}
\vdots & & & \\
F_{0,2} \arrow[u, hook] \arrow[r, hook] & F_{1,2} \arrow[r, hook] & F_{2,2} & \\
F_{0,1} \arrow[u, hook] \arrow[r, hook] & F_{1,1} \arrow[u, hook] \arrow[r, hook] & F_{2,1} \arrow[u, hook] \arrow[r, hook] & F_{3,1} \\
F_{0,0} \arrow[u, hook] \arrow[r, hook] & F_{1,0} \arrow[u, hook, "\text{inclusions}"'] \arrow[r] & F_{2,0} \arrow[u, hook] \arrow[r] & F_{3,0} \arrow[u, hook] \arrow[r] & \cdots
\end{tikzcd}
$$

commutative diagram

Bipersistence Module

Let M be a collection of vector spaces $\{M_a\}_{a \in \mathbb{Z}^2}$ with maps $\{M_{a,b} : M_a \to M_b\}_{a \le b}$

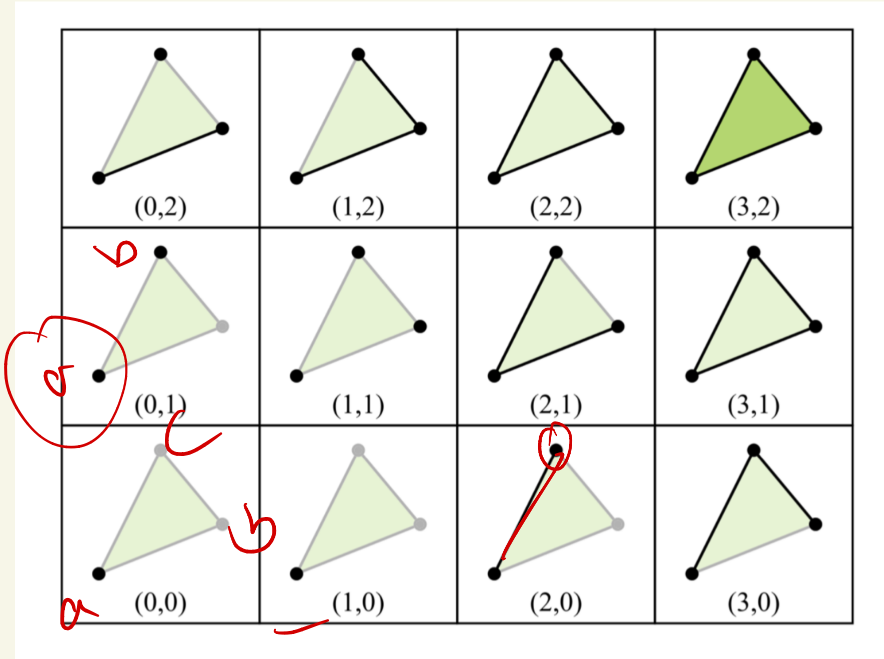
H_d

$$
\begin{tikzcd}
M_{0,2} \arrow[r] & M_{1,2} \arrow[r] & M_{2,2} \arrow[r, dashed] & {} \\
M_{0,1} \arrow[r] \arrow[u] & M_{1,1} \arrow[r] \arrow[u] & M_{2,1} \arrow[r, dashed] \arrow[u] & {} \\
M_{0,0} \arrow[r] \arrow[u] & M_{1,0} \arrow[r] \arrow[u] & M_{2,0} \arrow[u] & \cdots
\end{tikzcd}
$$

induce maps

linear maps b/t homology groups

Simple example



0-homology w/ k the field

← we've fixed $\mathbb{Z}_2$

$$\begin{tikzcd}
k^2 \arrow[r] & k \arrow[r] & k \arrow[r] & k \\
k^2 \arrow[u] \arrow[r] & k^3 \arrow[u] \arrow[r] & k \arrow[u] \arrow[r] & k \arrow[u] \\
k \arrow[u] \arrow[r, "id"] & k \arrow[u] \arrow[r] & k \arrow[u] \arrow[r] & k \arrow[u]
\end{tikzcd}$$

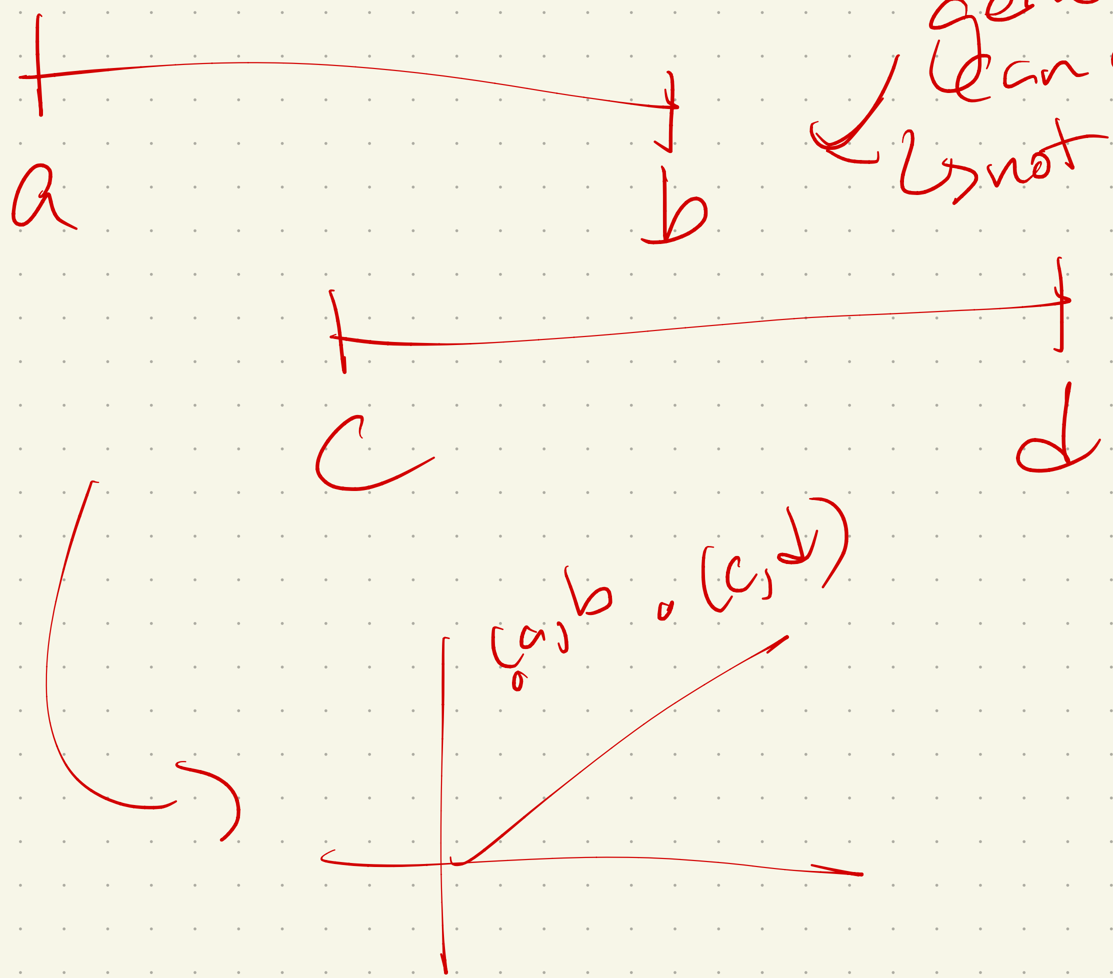
$\{a\}$ $\{a\}$ $\{a\}$

maps

Recall Barcodes/diagrams are unique representations, b/c of Gabriel's theorem:

$$H_d(M_0) \longrightarrow H_d(M_1) \longrightarrow \cdots \longrightarrow H_d(M_{n-1}) \longrightarrow H_d(M_n)$$

generators can change
↳ not start & end



If we consider the field $= k$, can look at maps explicitly:

$$\mathbf{k} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \mathbf{k}^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \mathbf{k} \xrightarrow{\begin{pmatrix}0\\1\end{pmatrix}} \mathbf{k}^2 \xrightarrow{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}} \mathbf{k}^2$$

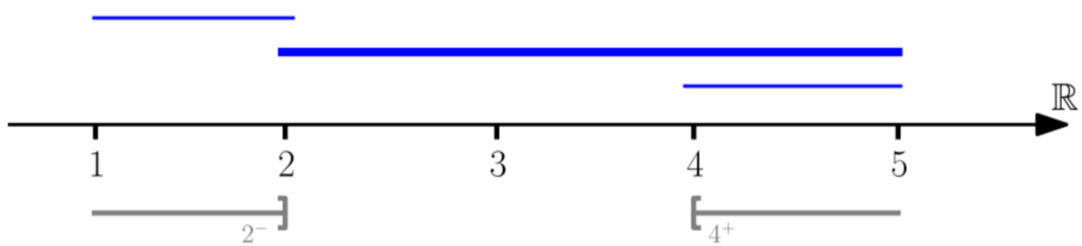


Fig. 1 A one-parameter persistence module M (top) indexed over $\{1, 2, 3, 4, 5\} \subset \mathbb{R}$, and the graphical representation of its barcode (in blue). The corresponding rank decomposition $\mathrm{Rk}\, M = \mathrm{Rk}\, \mathbf{k}_{[\![1,2]\!]} + \mathrm{Rk}\, \mathbf{k}_{[\![2,5]\!]} + \mathrm{Rk}\, \mathbf{k}_{[\![4,5]\!]}$ is readily available, and the ranks can easily be read from it: for instance, the rank $\mathrm{Rk}\, M(2, 4) = 1$ is given by the one bar (thickened) that connects the down-set 2^- to the up-set 4^+ (Color figure online)

figure from:
Botnan, Oppermann + Oudot
2025

The bad news

Theorem Carlsson & Zomorodian 2009
(paraphrased a bit)

The algebraic classification of indecomposables of multiparameter persistence modules contain both discrete & continuous portions

$\Longrightarrow$ no PD-diagram-like representation is possible.

(It is of "wild representation type")

Lowering the dimension

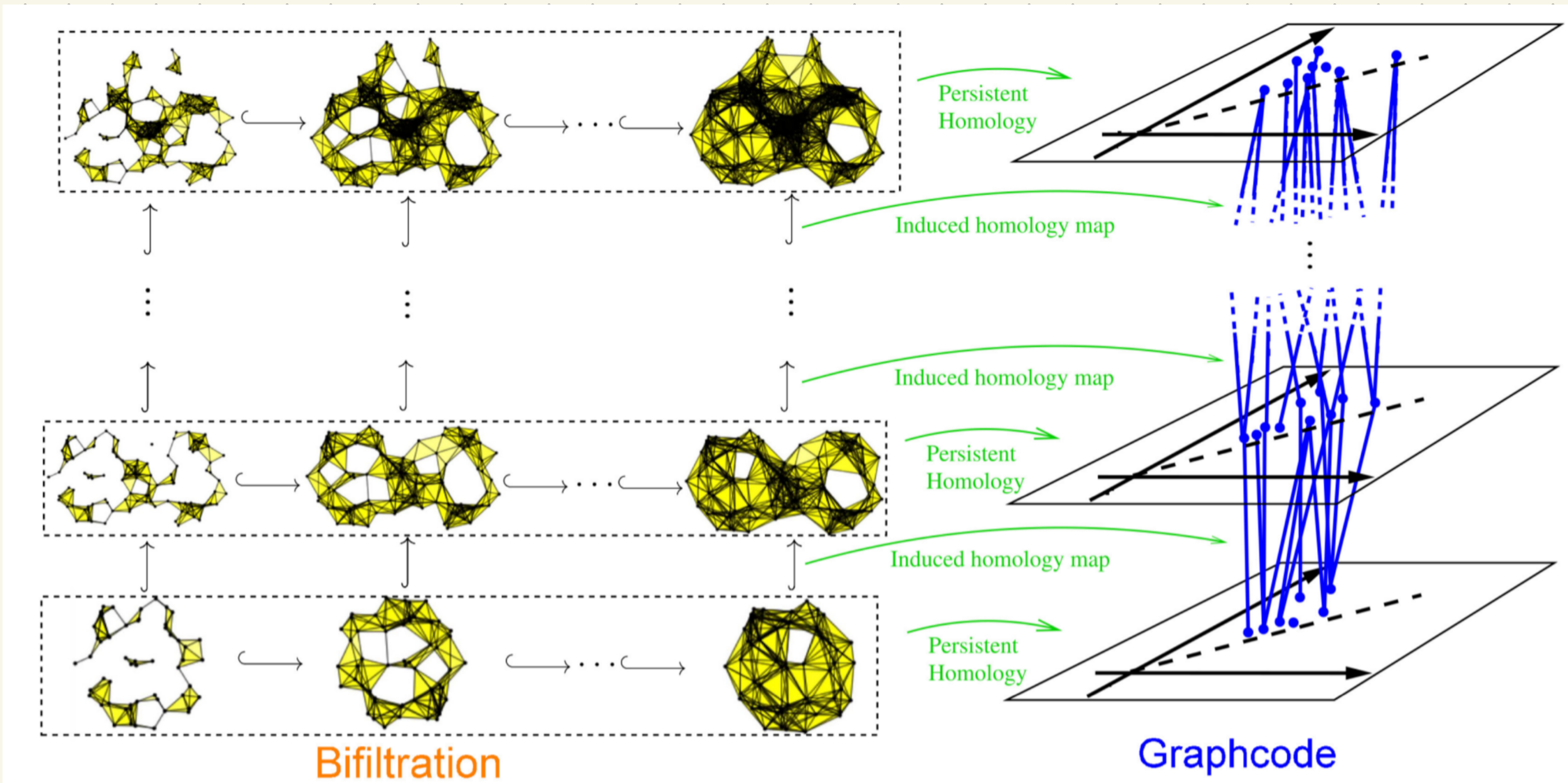
One common approach is to restrict to a "line":

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
M_{0,2} \arrow[r] \arrow[u] & M_{1,2} \arrow[r] \arrow[u] & M_{2,2} \arrow[r] \arrow[u] & M_{3,2} \arrow[r] \arrow[u] & \cdots \\
M_{0,1} \arrow[r] \arrow[u] & M_{1,1} \arrow[r] \arrow[u] & M_{2,1} \arrow[r] \arrow[u] & M_{3,1} \arrow[r] \arrow[u] & \cdots \\
M_{0,0} \arrow[r] \arrow[u] & M_{1,0} \arrow[r] \arrow[u] & M_{2,0} \arrow[r] \arrow[u] & M_{3,0} \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Forget about a lot!

$$M_{0,0} \longrightarrow M_{1,0} \longrightarrow M_{1,1} \longrightarrow M_{2,1} \longrightarrow M_{3,1} \longrightarrow \cdots$$

Graphcodes (for ML pipelines) $\boxed{\text{Kerber + Russold 2024}}$



Each PD is easy, but "vineyard" maps are a bit trickier.

Here:

$$\begin{tikzcd}
K_1 \arrow[r, hook] & K_2 \arrow[r, hook] & \cdots \arrow[r, hook] & K_n \\
L_1 \arrow[r, hook] \arrow[u, hook] & L_2 \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & L_n \arrow[u, hook]
\end{tikzcd}$$

Each inclusion $L_i \hookrightarrow K_i$ extends to a map on chain complexes (& hence cycles):

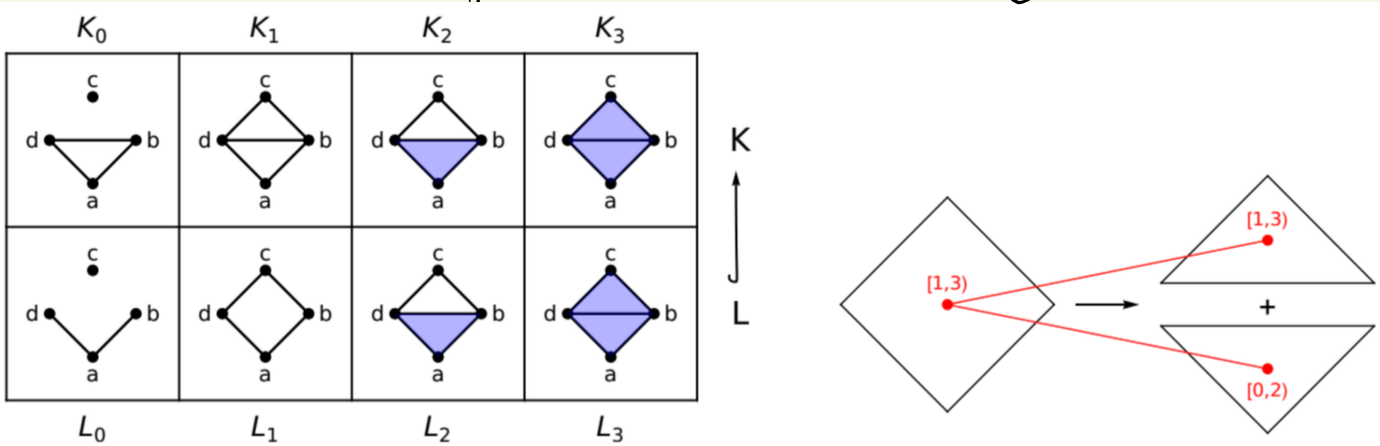


Figure 3: Left, lower row: $Z_1(L)$ is generated by the cycles $abcd$ and abd. They form a barcode basis, with attached bars $[1,3)$ and $[2,2)$, respectively. Note that also abd and bcd form a basis of $Z_1(L)$, but that is not a barcode basis as none of these cycles is already born at L_1, so they do not induce a basis of $Z_1(L_1)$. Left, upper row: Here, abd and bcd form a barcode basis with attached bars $[0,2)$ and $[1,3)$, respectively, and abd and $abcd$ as well (with identical barcode).
Right: Choosing the basis $abcd$, abd for $Z_1(L)$ and abd and bcd for $Z_1(K)$, we have $abcd = abd + bcd$, hence the cycle $abcd$ has two outgoing edges, to both basis elements in K. We ignore the basis vector abd of L in the figure, since its birth and death index coincide, so the corresponding

Next time: Dr. Danny Chen

Guest lecture: TDA in medical imaging

Friday:

Representation theory, + rank invariants!

(Will get mathier.)

Now:

Project proposals + time to discuss