TDA - fall 2025

Morse theory
Simplicial
Complexes

Where were we...

- Basic dfns: open sets, topological space, maps $f: X \to Y$
- Ways to be "the same":
  - homeomorphism
  - isotopy
  - ambient isotopy
  - homotopic
  - homotopy equivalent
  - deformation retract

all about existence of maps

Manifolds

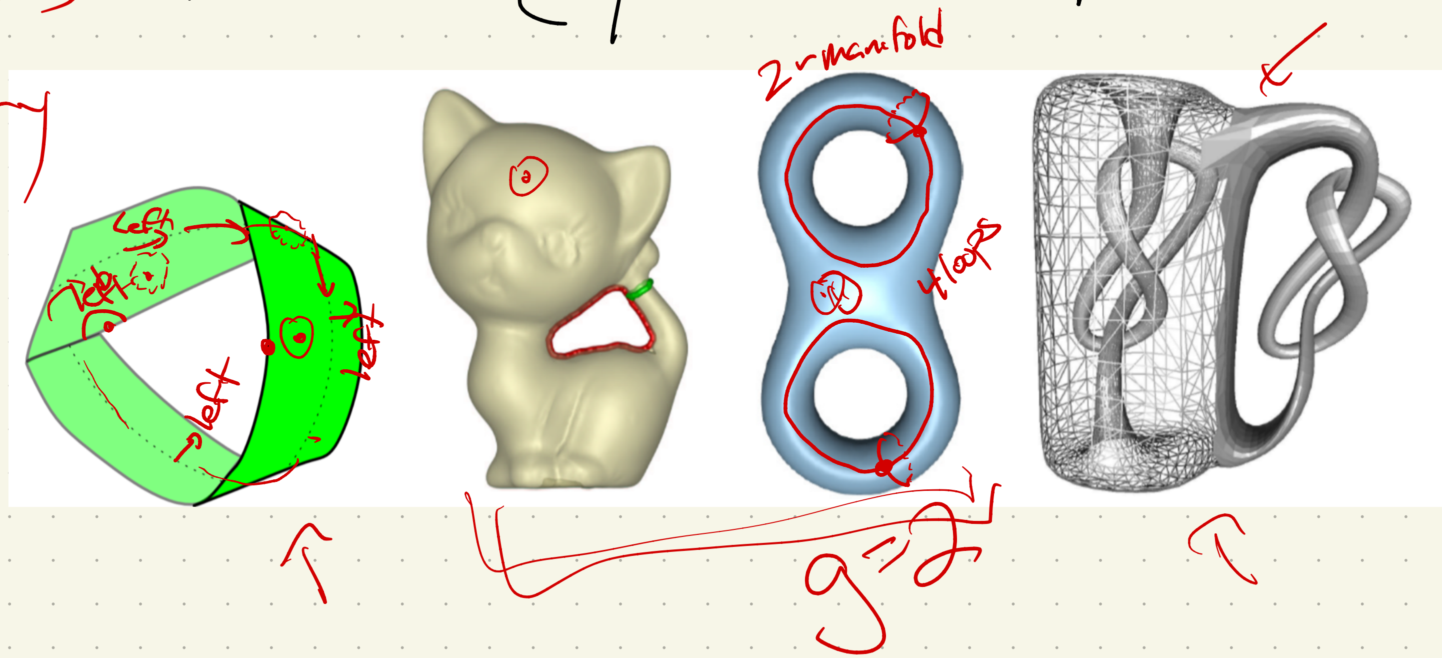
A topological space is an m-manifold if every $x \in M$ has a point homeomorphic to the m-ball $\mathbb{B}_0^m$ or the m-halfspace $\mathbb{H}^m$:

$\mathbb{B}_0^2$

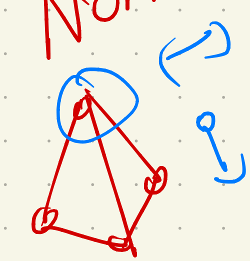
$$\mathbb{B}_0^m = \{y \in \mathbb{R}^m \mid \|y\| < 1\}$$

$$\mathbb{H}^m = \{y \in \mathbb{R}^m \mid \|y\| < 1 \text{ and } y_m \geq 0\}$$

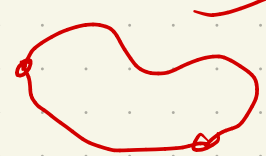
Boundary



Non:



Notation / terminology

- boundary : look like $\mathbb{H}^d$
- Surface : 2-manifold
- Non-orientable : walk along a curve starting on one side. If you could end up on other side when you return → non-orientable
- Loop : 1-manifold, no boundary $\mathbb{R}$ 
- Genus g : $\exists$ a set of $2g$ loops which can be removed without disconnecting it.

Smooth

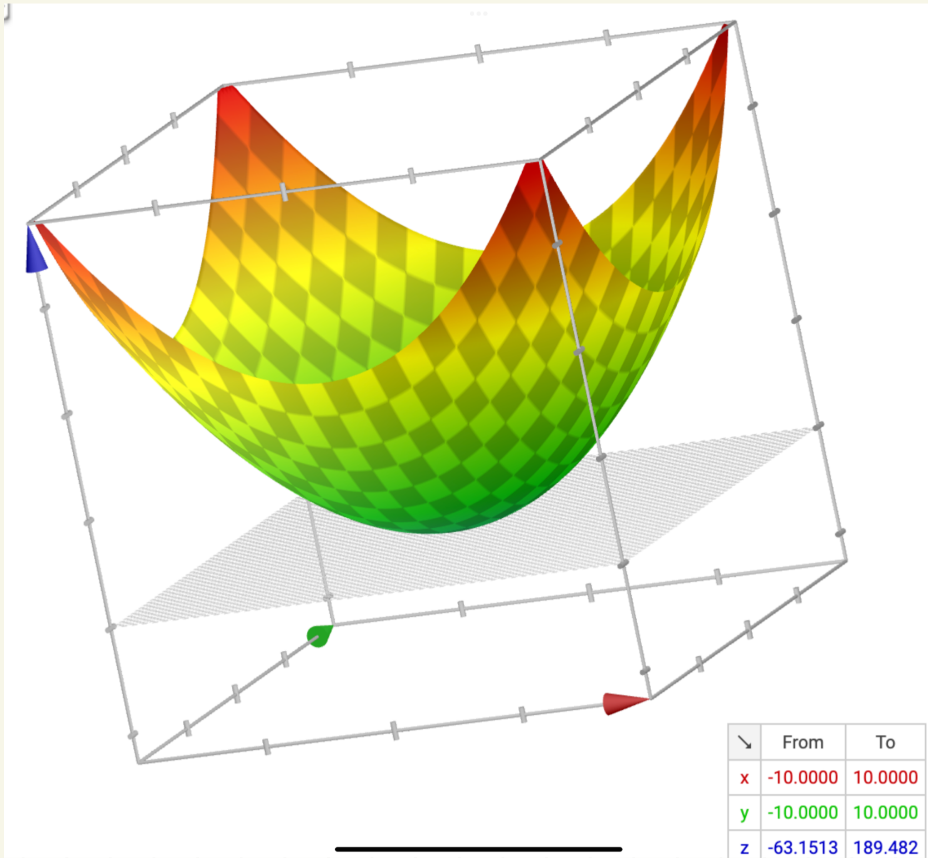
Topological manifolds are spaces.
But usually, consider an embedding into Euclidean space $\Rightarrow$ geometry.

Given a smooth function $f: \mathbb{R}^d \to \mathbb{R}$, the gradient vector field $\nabla f: \mathbb{R}^d \to \mathbb{R}^d$ at a point x is:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right]$$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f((x_1, x_2)) = x_1^2 + x_2^2$$



$$\nabla f =$$

$$\left[\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f\right]$$

$$= [2x_1, 2x_2]$$

Then $\nabla f((0,0)) = [0,0]$

$\nabla f((1,0)) = [2,1]$

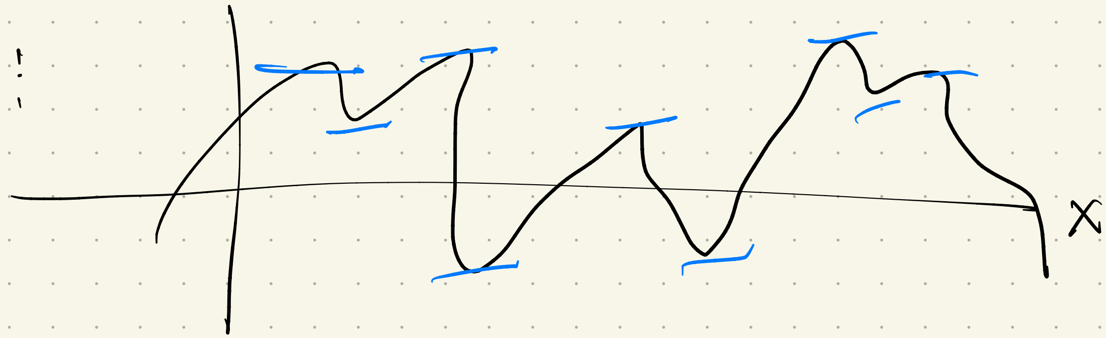
Critical point

Any $p \in \mathbb{R}^d$ where $\nabla f(p) = \vec{0}$

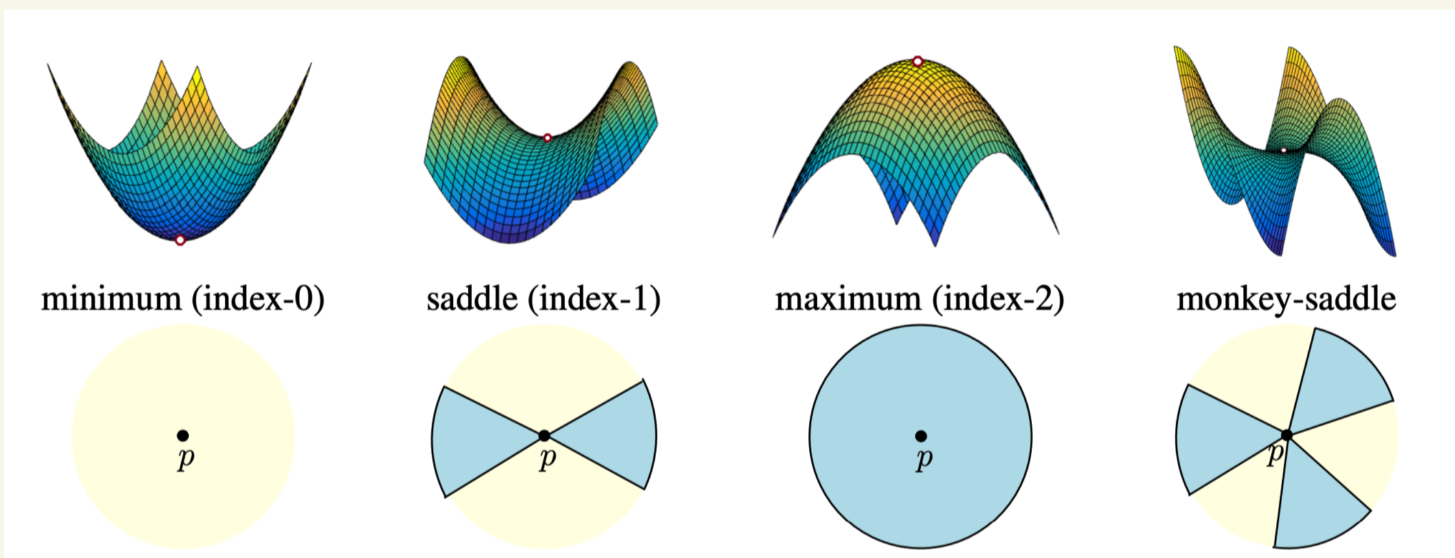
(Otherwise we say p is regular)

On 1 manifolds:

$$\frac{\partial f}{\partial x} \cdot x = 0$$



On 2 manifolds.



Extending to manifolds:

Given $\Phi: U \to W$, $U \subseteq \mathbb{R}^k$ & $W \subseteq \mathbb{R}^d$ open sets, where

$$\Phi(x) = (\phi_1(x), \ldots, \phi_d(x))$$

The Jacobian of Φ is a $d \times k$ matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial \phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \phi_1(x)}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_d(x)}{\partial x_1} & \cdots & \frac{\partial \phi_d(x)}{\partial x_k} \end{bmatrix}$$

Types of critical points

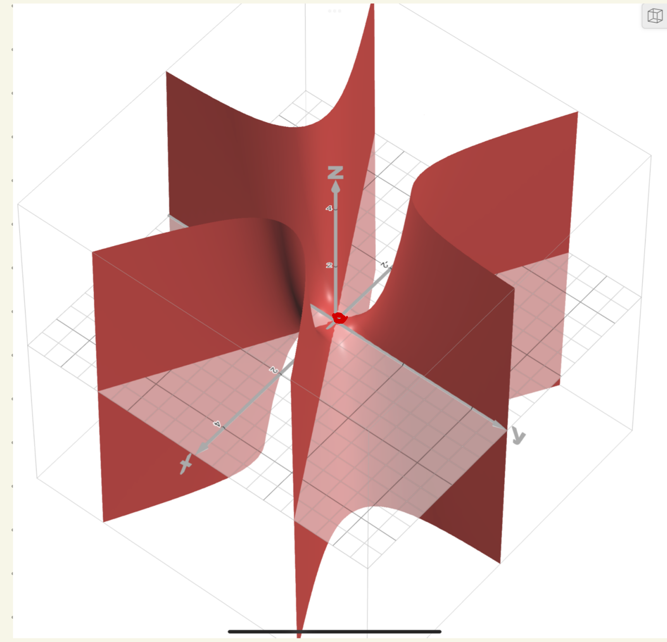
For a smooth m-manifold, the Hessian matrix of $f: M \to \mathbb{R}$ is the matrix of 2^{nd} order partial derivatives:

$$Hessian(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix}$$

A critical point is non-degenerate if Hessian is nonsingular ($\det \neq 0$); otherwise degenerate.

An example: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^3 - 3x_1x_2^2$$



$$\nabla f = \left[3x_1^2 - 3x_2^2, \; -6x_1x_2\right]$$

$$\overset{(0,0)}{\Longrightarrow} [0, 0]$$

Is it degenerate?

Hessian:

$$\begin{pmatrix} \frac{\partial}{\partial x_1 \partial x_1} & \frac{\partial}{\partial x_1 \partial x_2} \\ \frac{\partial}{\partial x_2 \partial x_1} & \frac{\partial}{\partial x_2 \partial x_2} \end{pmatrix} = \begin{bmatrix} 6x_1 & -6x_2 \\ -6x_2 & -6x_1 \end{bmatrix}$$

So at $(0,0)$, $\det = 0$

degenerate & critical at $(0,0)$

Morse Lemma

Given a smooth function $f: M \to \mathbb{R}$ defined on a smooth manifold M, let p be a non-degenerate critical point of f. Then $\exists$ a local coordinate system in a neighborhood $U(p)$ s.t.

- p's coordinanate is $\vec{0}$
- locally, any x is in the form

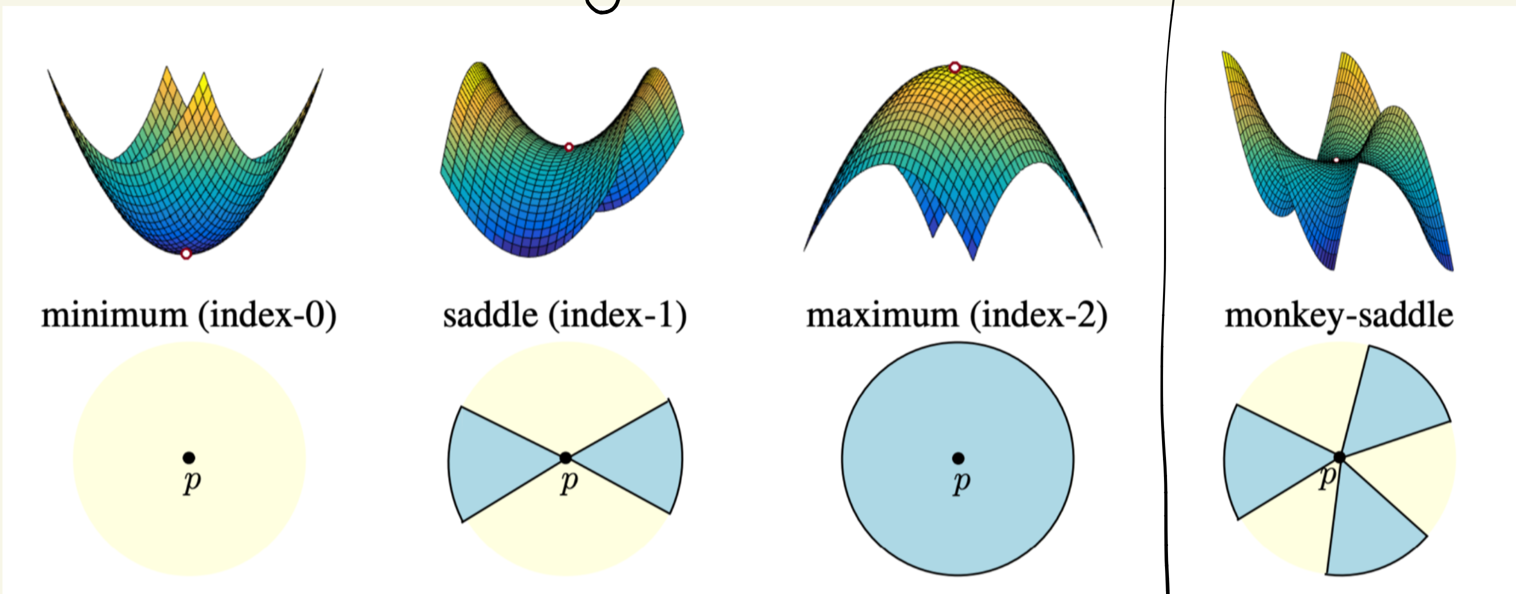
$$f(x) = f(p) - x_1^2 - \cdots - x_s^2 + x_{s+1}^2 + \cdots + x_m^2$$

for some $s \in [0, m]$

s is called the **index** of p.

Back to that picture...

non-degenerate | degenerate



everything is bigger around p

one coordinate bigger, one smaller

everything is smaller around p

Morse functions

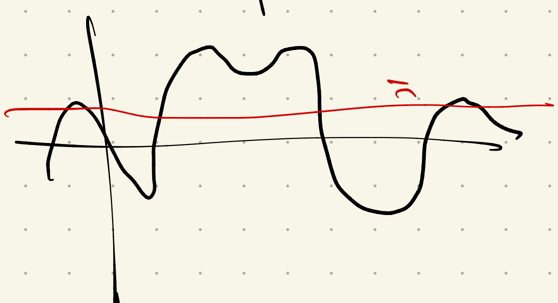
A smooth function $f: M \to \mathbb{R}$ (on a smooth manifold M) is a Morse function if

- none of f's critical points are degenerate
- the critical points have distinct function values

Some examples: Morse?

$f: \mathbb{R} \to \mathbb{R}$

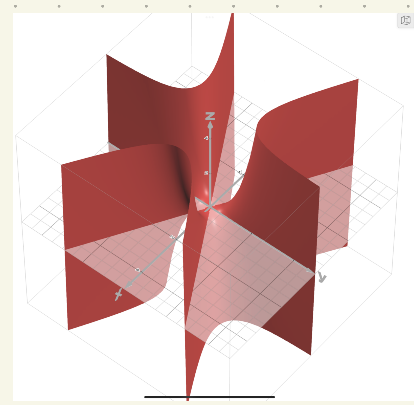
NO



NO

$g: \mathbb{R}^2 \to \mathbb{R}$

$g((x_1, x_2)) = x_1^3 - 3x_1x_2^2$

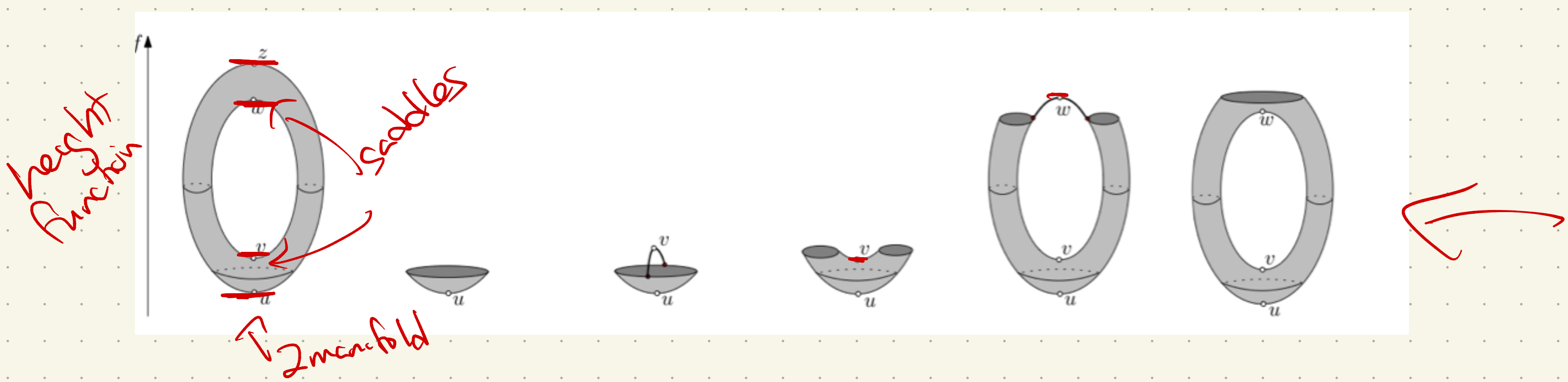


Why should we care??

Looking ahead:

- often won't just have a space, but also some measurements
  $\Rightarrow$ a function!
- Every function (almost) is Morse

In TDA: many signatures study how the function changes $\longrightarrow$ level sets



Level sets

Given $f: M \to \mathbb{R}$, the interval level set for f with respect to $I \subseteq \mathbb{R}$ is

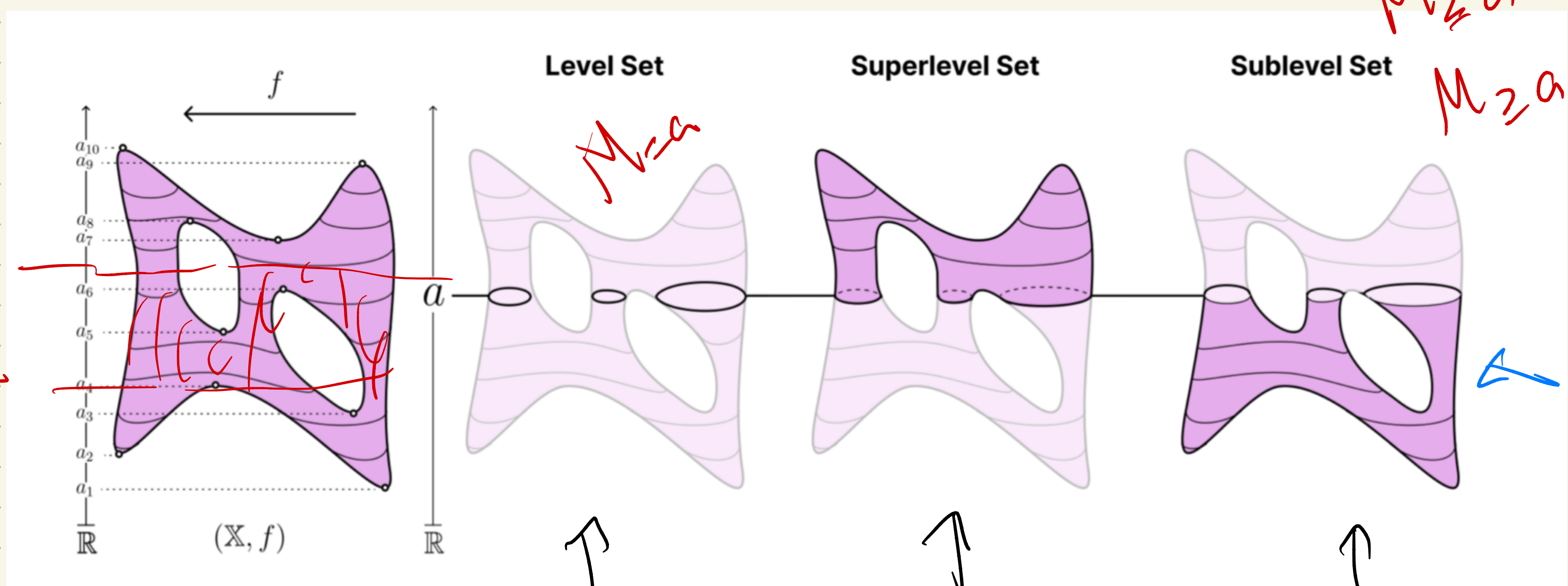
$$M_I := f^{-1}(I) = \{x \in M \mid f(x) \in I\}$$

Special types of intervals:

$M_{[a,b]}$

$M_{\leq a}$

$M_{\geq a}$



I

$M_{=a}$

$M_{\leq a}$

$I = [a]$ $\quad$ $I = [a, \infty]$ $\quad$ $I = [-\infty, a]$

What is the topology?

homeomorphism on differentiable spaces: invertible map where function & inverse are differentiable (& not just continuous)

Theorem 1.3 (Homotopy type of sublevel sets)**.** *Let $f : M \to \mathbb{R}$ be a smooth function defined on a manifold M. Given $a < b$, suppose the interval levelset $M_{[a,b]} = f^{-1}([a,b])$ is compact and contains no critical points of f. Then $M_{\leq a}$ is diffeomorphic to $M_{\leq b}$.*

Furthermore, $M_{\leq a}$ is a deformation retract of $M_{\leq b}$, and the inclusion map $i : M_{\leq a} \hookrightarrow M_{\leq b}$ is a homotopy equivalence.

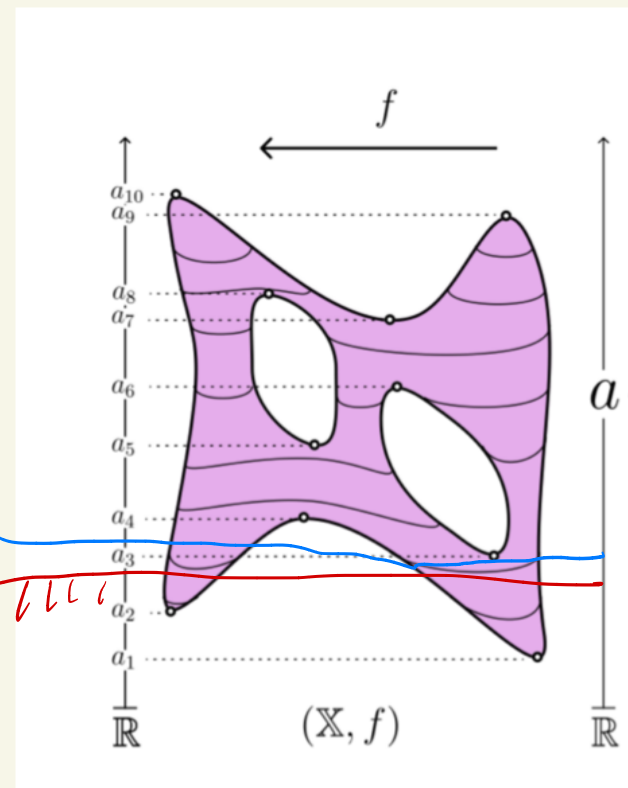
What does this mean?

Nothing interesting in between critical values

b

a

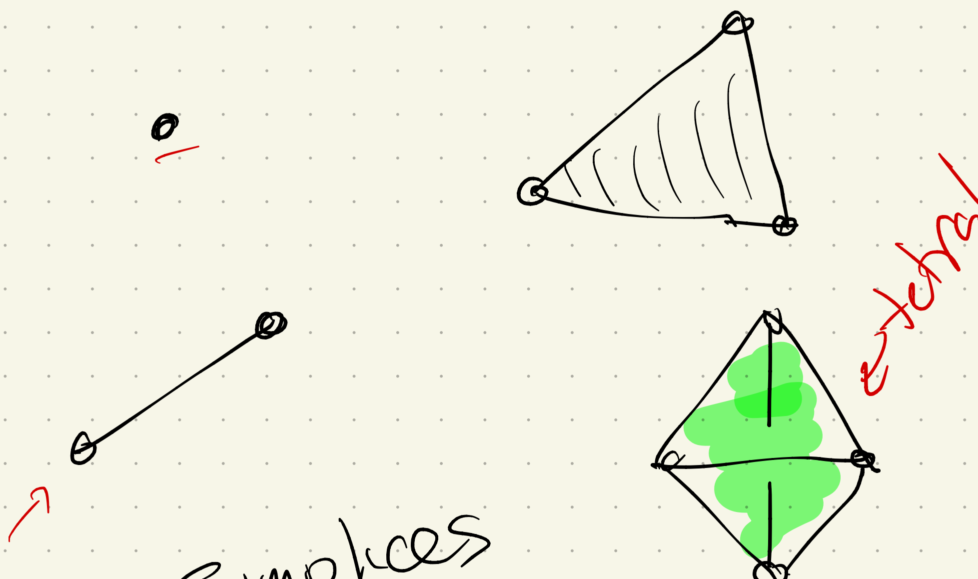
$[-\infty, a]$



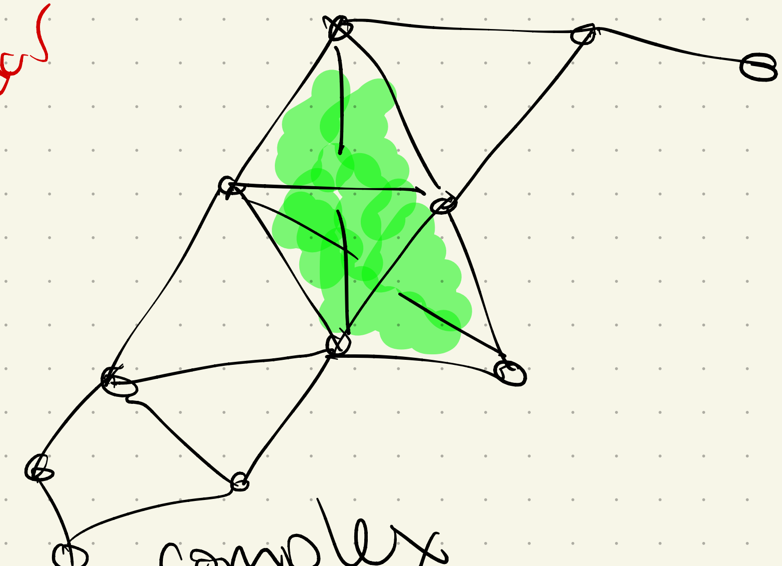
Simplicial Complexes

Computation requires a method to store data ⟶ discretely (usually)

A simplicial complex is a natural generalization of a graph:



Simplices



complex

More formally:

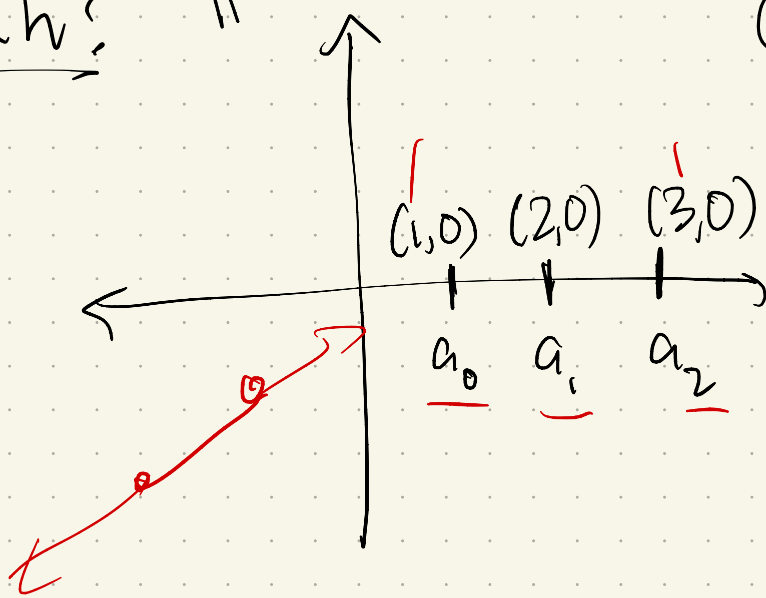
A set $\{a_0, \dots, a_k\} \subset \mathbb{R}^m$ is affinely independent if $\forall \{t_i\}_{i=0}^{k}$, the

equations $\sum_{i=0}^{k} t_i = 1$ and $\sum_{i=0}^{k} t_i a_i = 0$

$a_i - a_0, \dots, a_k - a_0$

$\Longrightarrow t_i = 0 \quad \forall i.$

... huh? $\mathbb{R}^2$



Q: can we find t_0, t_1, t_2

s.t. $t_0 + t_1 + t_2 = 1$

and $t_0 a_0 + t_1 a_1 + t_2 a_2 = 0$?

$t_0 = 1$ $\quad t_1 = -2$

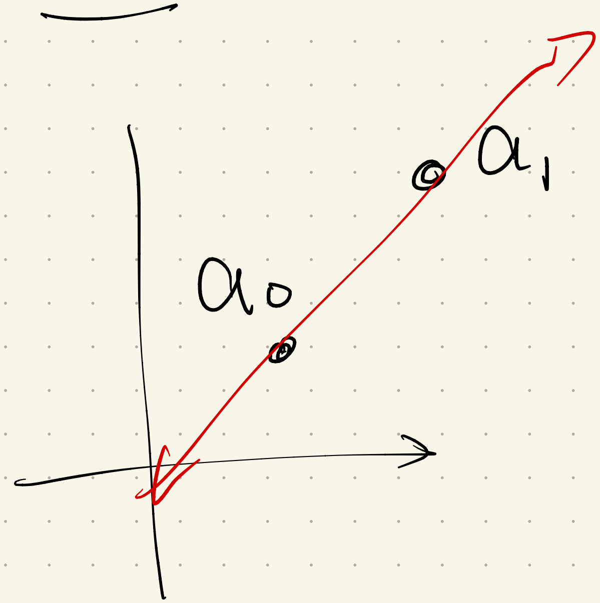
$t_2 = 1$

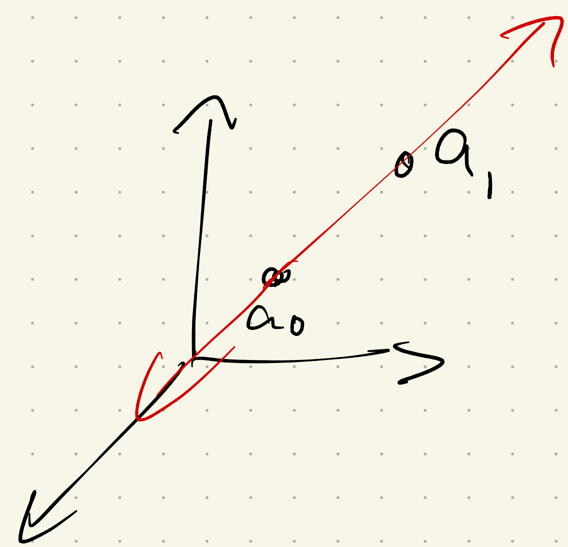
$(1,0) - 2\cdot(2,0) + (3,0) = (0,0)$

Given a set of affinely independent points $\{a_0, \dots, a_k\}$, the k-plane P spanned by the points is

$$P = \left\{ \sum_{i=0} t_i a_i \in \mathbb{R} \;\middle|\; \sum t_i = 1 \right\}$$

Note:

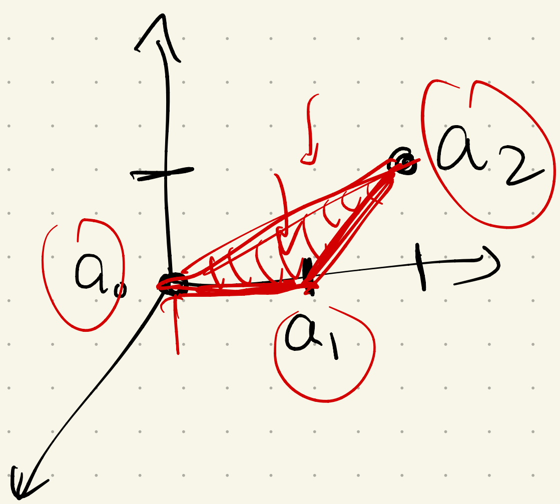




Given a set of K affinely independent points $\{a_0, \ldots, a_K\}$, the K-simplex σ spanned by the points is

$$P = \left\{ \sum_{i=0}^{K} t_i a_i \in \mathbb{R}^N \;\middle|\; \sum t_i = 1, \text{ \& } \forall i, t_i \geq 0 \right\}$$

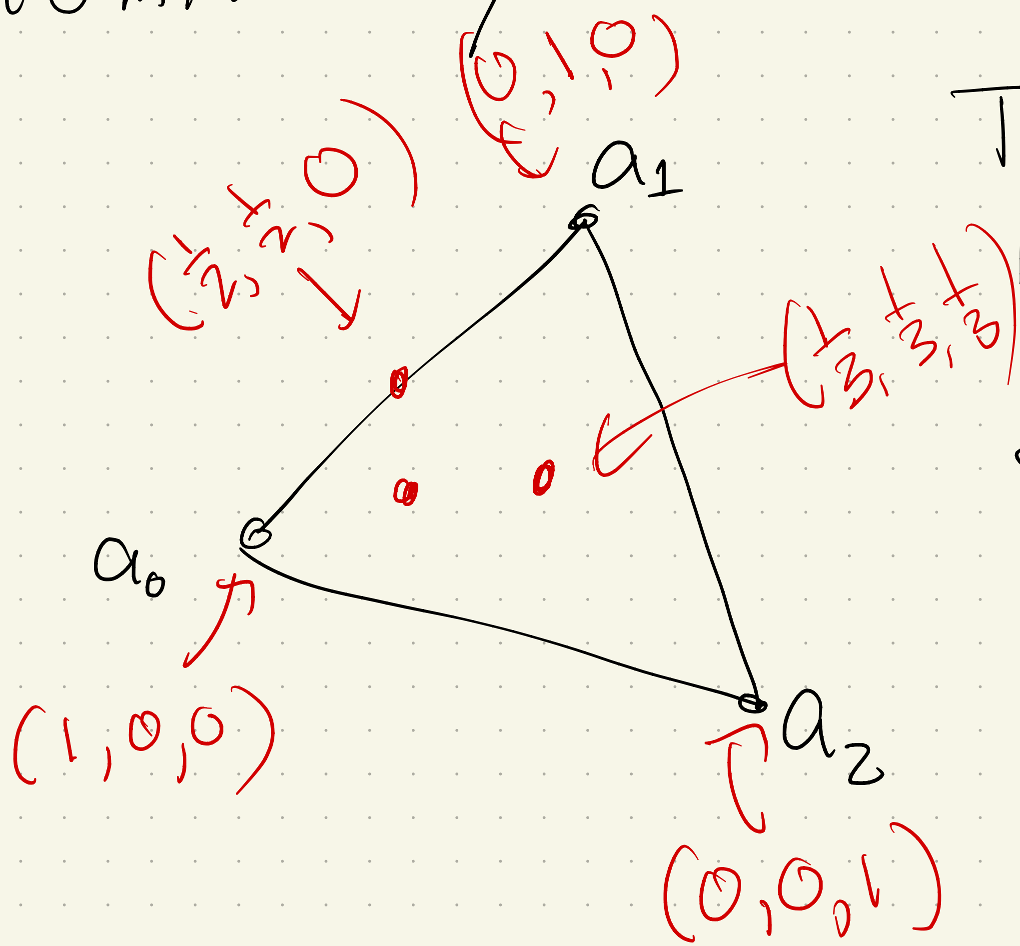
Example: in $\mathbb{R}^3$



$a_0 = (0,0,0)$
$a_1 = (1,0,0)$
$a_2 = (2,1,0)$

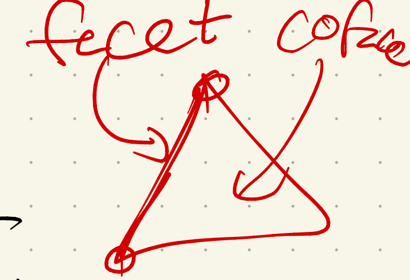
Barycentric coordinates

Fix $\{a_0, \dots, a_k\}$ and some $x \in$ k-simplex. Then the numbers $t_0, \dots, t_k$ are uniquely determined by x.

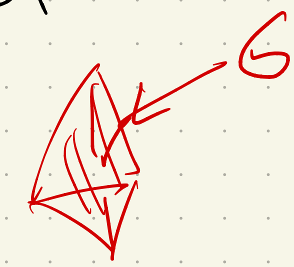


The barycenter is the point given by the coordinates $\left(\frac{1}{k+1}, \dots, \frac{1}{k+1}\right)$

Definitions



- $\{a_0, \ldots, a_k\}$ are the vertices of σ.
- The dimension of $\sigma = [a_0, \ldots, a_k]$ is k.
- Any simplex spanned by a subset of $\{a_0, \ldots, a_k\}$ is a face of σ



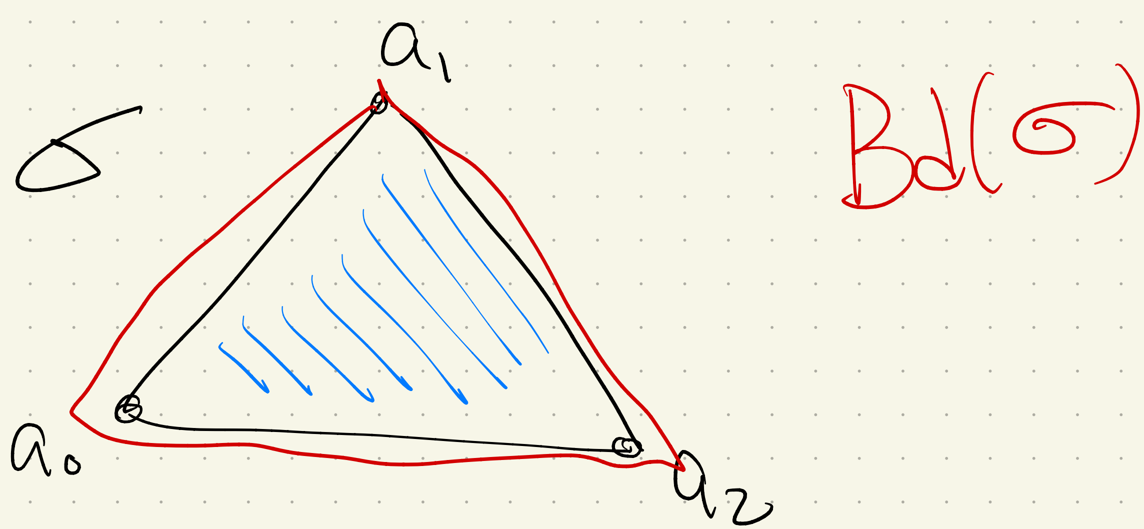
↳ proper face if $\neq \sigma$

↳ σ is a co-face of any of its faces

↳ if face has $\dim = k-1$, called a facet

Definitions (cont)

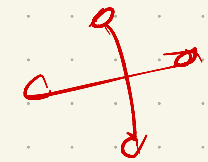
- The union of proper faces is the **boundary** of σ, $Bd(\sigma)$
- The **interior** of σ is $\sigma - Bd(\sigma)$
  - ↳ called open simplex

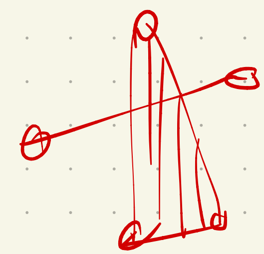


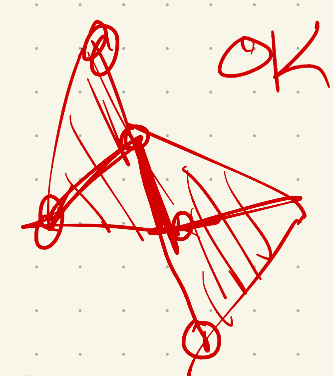
Simplicial Complex (Embedded or geometric)

A simplicial complex $K \subset \mathbb{R}^n$ is a (finite) collection of simplices in $\mathbb{R}^n$ s.t.

- every face of a simplex $\sigma \in K$ is also in K
- $\forall \sigma, \tau \in K$, $\sigma \cap \tau \in K$

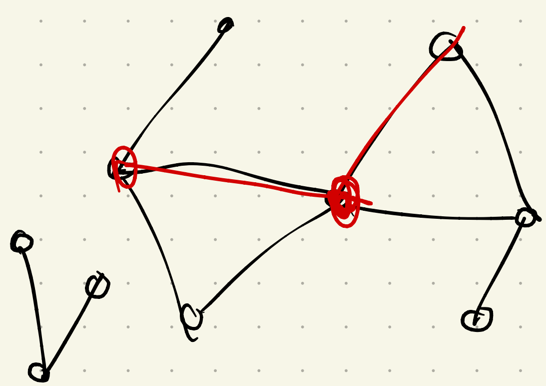


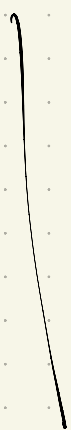


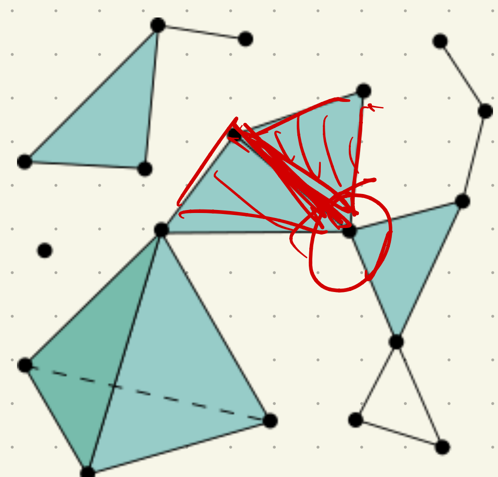


dimension of K $= \max_{\sigma \in K} \{\dim(\sigma)\}$

Examples







Note: Abstract simplicial complex K

a (finite) collection of (finite) non-empty subsets of a set $V = \{v_0, \dots, v_n\}$ s.t.

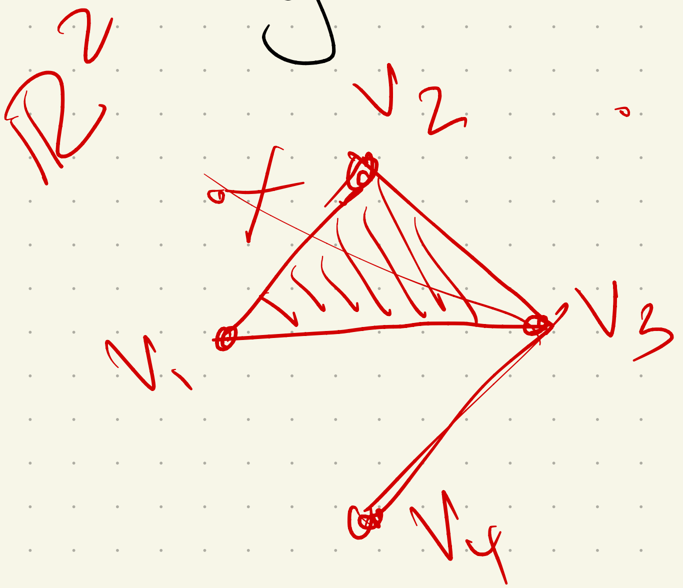
$\sigma \in K$ and $\tau \subseteq \sigma \Rightarrow \tau \in K$

Difference:

realization - not always easy!!

geometric

abstract



$V = \{v_1, v_2, v_3, v_4\}$

$K = \{ \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\},$
$\{v_1 v_2\}, \{v_2 v_3\}, \{v_1 v_3\},$
$\{v_3 v_4\}, \underbrace{\{v_1 v_2 v_3\}}_{2} \}$

forget embedding

- Geometric realizations of abstract simplicial complexes are <u>not</u> unique
  ↳ often write $|K|$ vs K
- In fact, computing embeddings in some $\mathbb{R}^n$ is a huge area of study
  - smallest $\mathbb{R}^n$ if K has dim d is classical topology
    Famous theorem: if $\dim(K)=k$, $\mathbb{R}^{2k+1}$ possible
  - On the other end, computing "nice" embeddings of graphs is a huge area of study

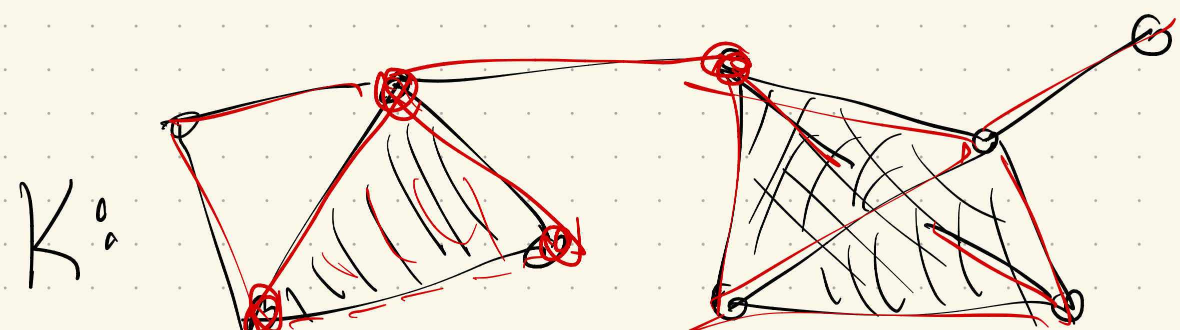
Subcomplexes & skeletons

If L is a subcollection of K that contains all faces of its elements, then L is a subcomplex.

A subcomplex is full if it has all simplices from K which are spanned by vertices in L.

The subcomplex of K containing all the simplices σ with $\dim(\sigma) \leq p$ is the p-skeleton.

1-skeleton: graph



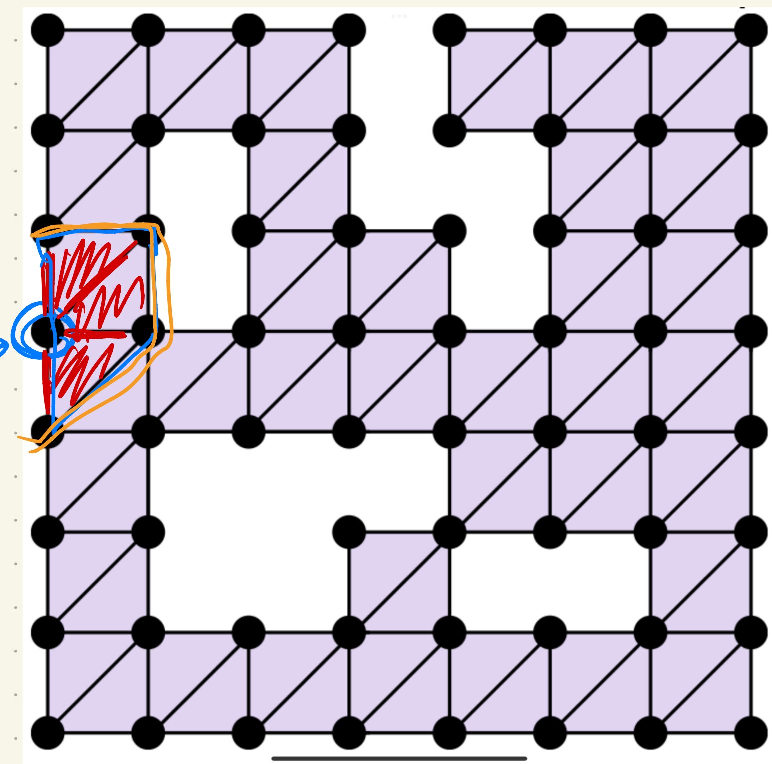
Stars & Links

The **star** of $\tau \in K$, $St(\tau) = \{\sigma \in K \mid \tau \leq \sigma\}$

(Warning: $st(\tau)$ is not a simplicial complex.)

$st(\tau)$

$\tau = \{v\}$



The **closed star** $\overline{St(\tau)}$ is the closure of $St(\tau)$.

The **link** of τ is $\overline{St(\tau)} - st(\tau)$
$= Lk(\tau)$

Triangulations

We say a simplicial complex K is a **triangulation** of a manifold M if the underlying space $|K|$ is homeomorphic to M.

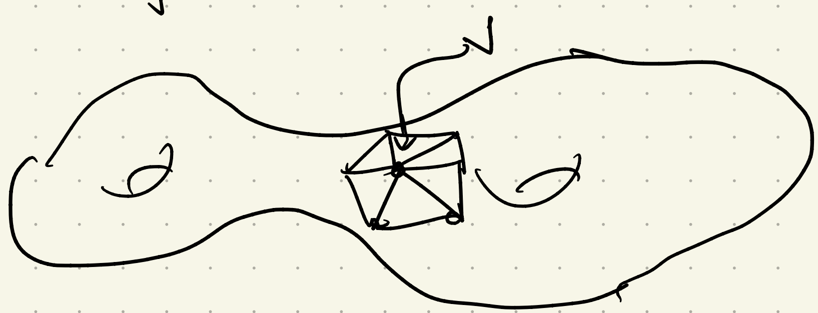
Note: If M is a k-manifold, $\dim(K)$ must be k also.

Useful facts:

$\forall v \in K$, $\quad |St(v)| \cong \mathbb{B}_0^k$ or $\mathbb{H}_0^k$

and $\quad |Lk(v)| \cong S^{k-1}$ or $\mathbb{B}_0^{k-1}$

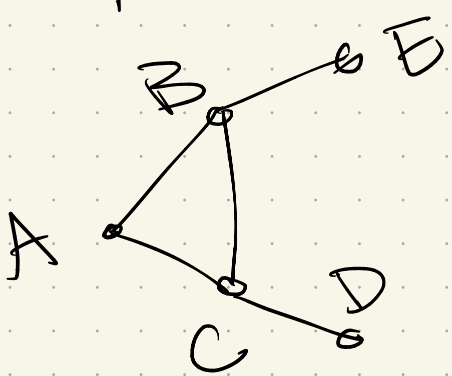
Ex: $\dim(K) = 2$

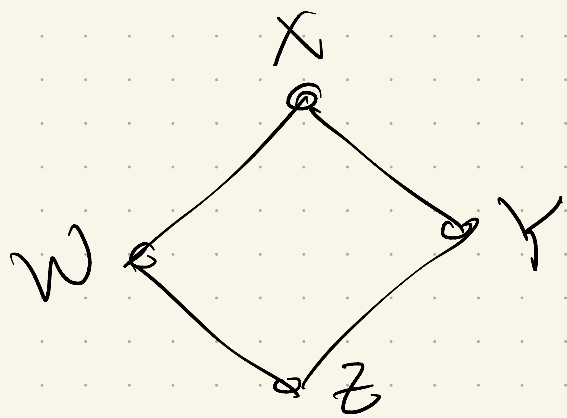


Simplicial maps

A map $f: K_1 \to K_2$ is called simplicial if $\forall\, \tau = \{v_0, \dots, v_k\} \in K_1$, we have the simplex $f(\tau) = \{f(v_0), \dots, f(v_k)\} \in K_2$

Example: Simplicial?





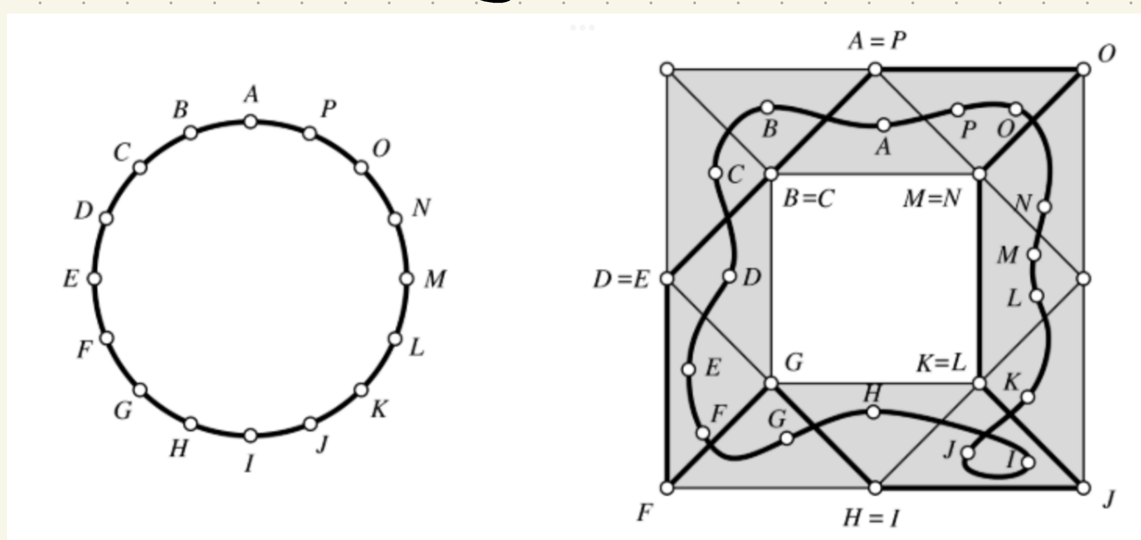
ℓ_1: $A \mapsto W$, $B \mapsto X$, $C \mapsto X$, $D \mapsto Y$, $E \mapsto Y$

ℓ_2: $A \mapsto X$, $B \mapsto Y$, $C \mapsto W$, $D \mapsto Z$, $E \mapsto Z$

Fact: Every continuous function $g: |K_1| \to |K_2|$ can be approximated by a simplicial map f on appropriate subdivisions of K_1 & K_2.

Here: for a point $x \in |K_1|$, $f(x)$ belongs to the minimal closed simplex $\sigma \in L_2$ that contains $g(x)$

Two maps shown: continuous g & simplicial f



Point clouds

Let X be a finite point set in a metric space (M,d).

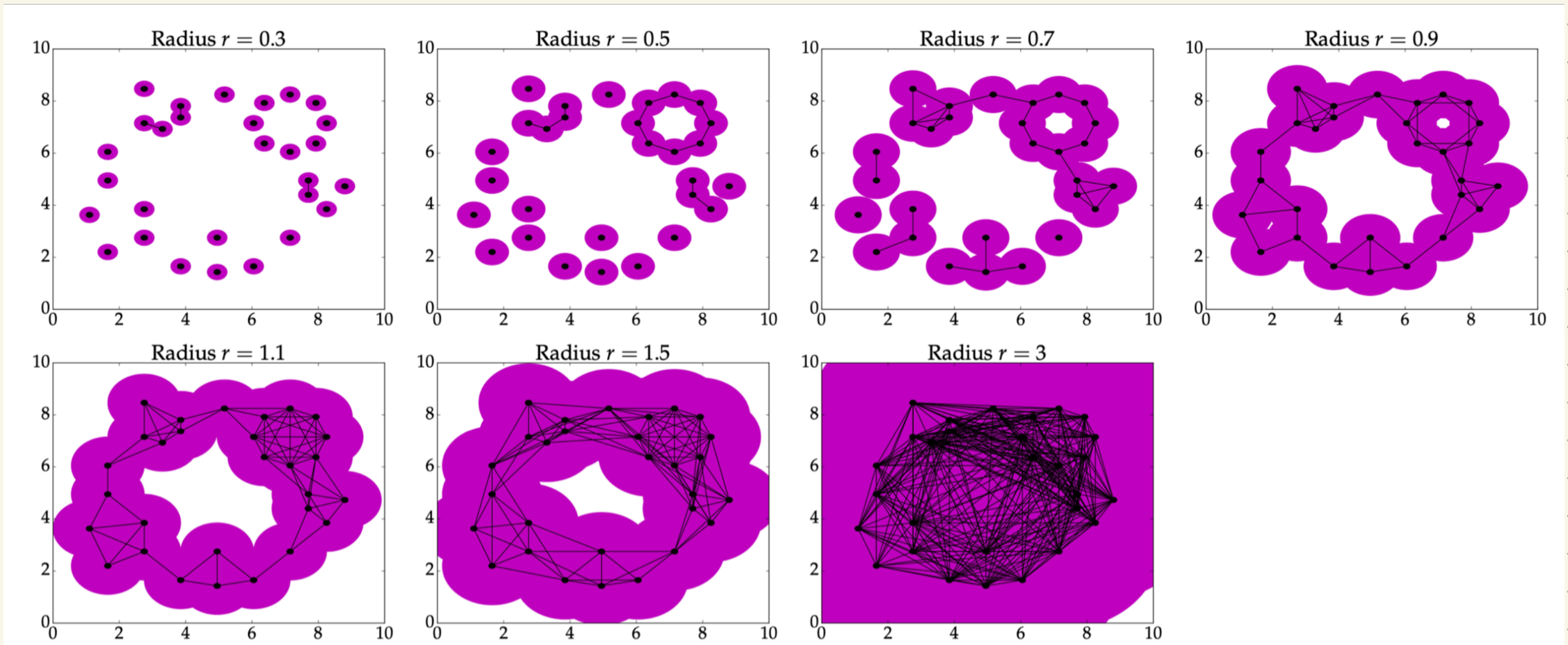
$\hookrightarrow$ often $(\mathbb{R}^d, \ell_2)$

Note: topology is pretty boring!

Let $B(x,r) = \{y \in M \mid d(x,y) \leq r\}$

(so these are closed)

Goal: Study how these balls interact.

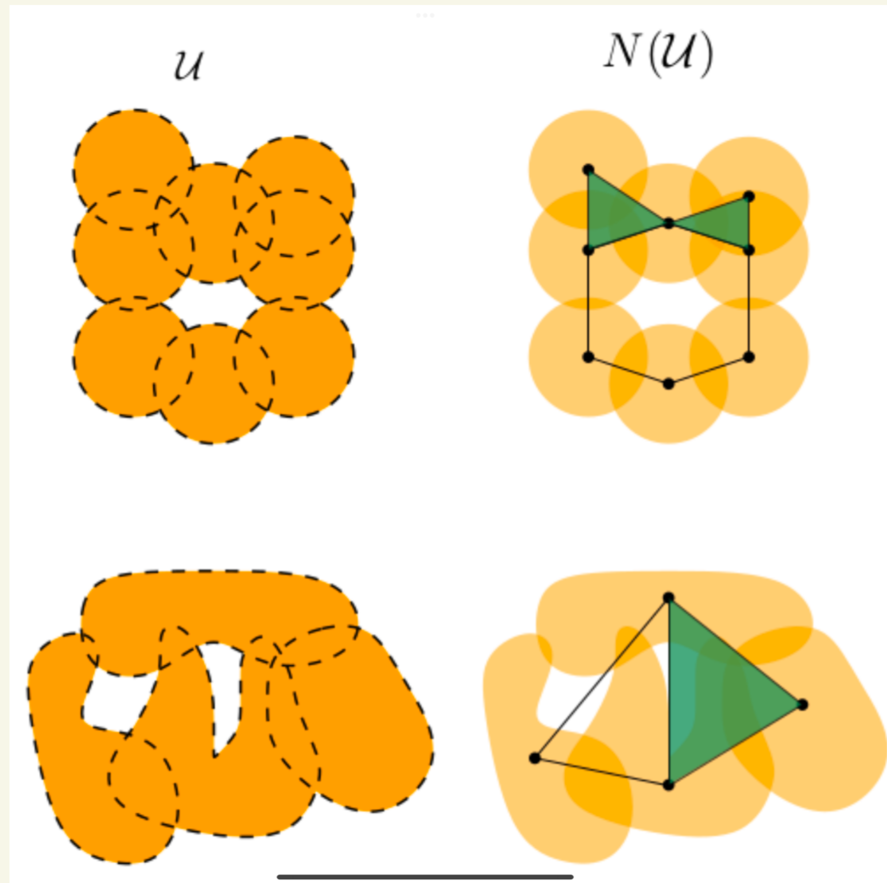


Note: there isn't a single correct r!

Given a finite collection of sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, the nerve of $\mathcal{U}$, $N(\mathcal{U})$, is the simplicial complex with vertex set A, where $\{\alpha_0, \ldots, \alpha_k\} \subseteq A$ is a k-simplex $\in N(\mathcal{U})$

$$\Longleftrightarrow$$

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha(k)} \neq \emptyset$$



Check: This is an abstract simplicial complex.

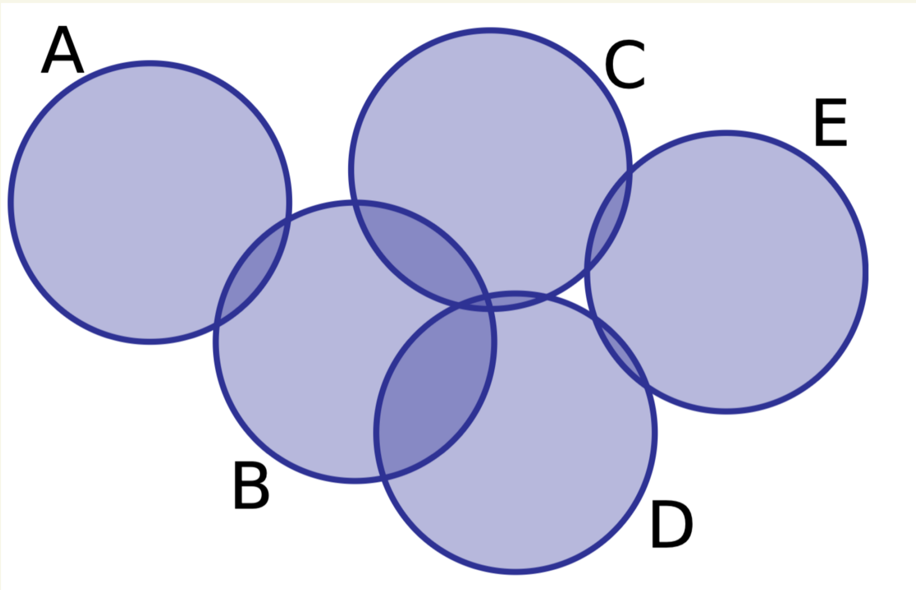
Need if $\sigma \in K$ + $\tau \leq \sigma$

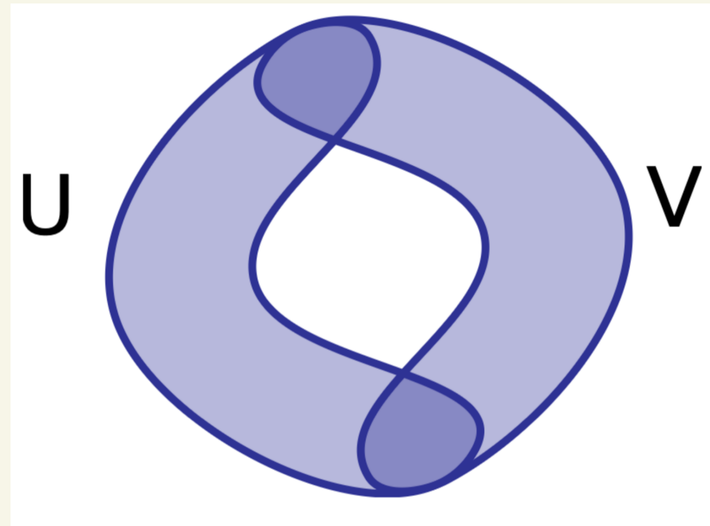
$\Rightarrow \tau \in K$

Here: if $\sigma = \{\alpha_0, \dots, \alpha_k\}$

$\Rightarrow$

Some examples to try:





Difference:

Nerve Lemma

Given a finite cover $\mathcal{U}$ (open or closed) of a metric space M, the underlying space $|N(\mathcal{U})|$ is homotopy equivalent to M if every non-empty intersection $\bigcap_{i=0}^{k} U_{\alpha_i}$ of cover elements is homotopic to a point (i.e. is contractible).

Why we care: