TDA - fall 2025

Monodromy

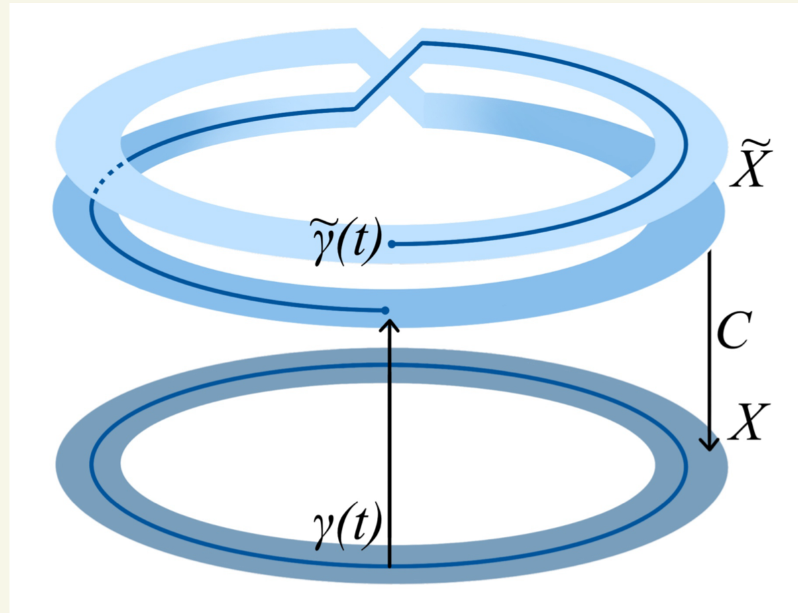
Last time: Directional Transforms

Take $A \subseteq \mathbb{R}^d$, & compute topological signature (ECC, PD, Reeb graph) for sublevelset filtration for $\underbrace{\text{every}}_{?}$ direction $\omega \in S^{d-1}$.

Big result/surprise:

Monodromy: effect where loops in a base space don't lift to loops in a covering space

Consider space X, loop γ, covering $\tilde{X}$, and a lift $\tilde{\gamma}$.



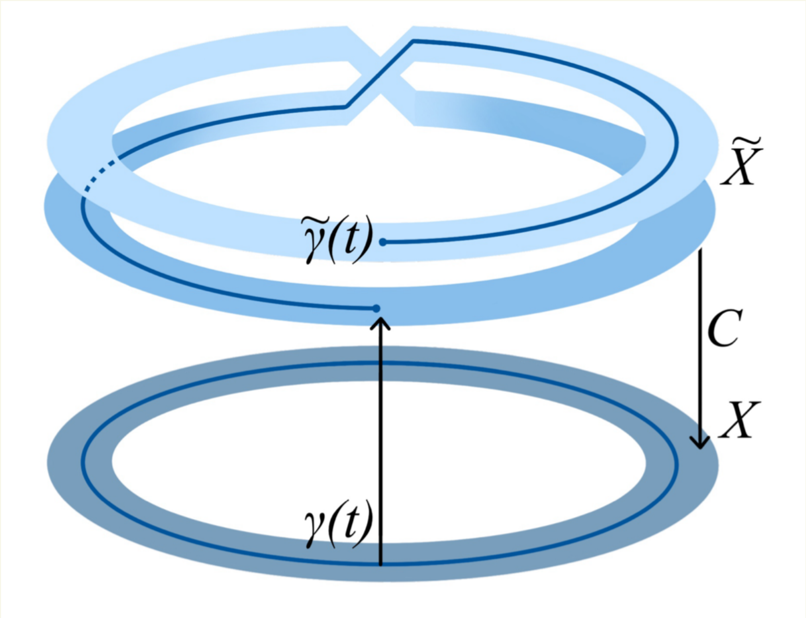
If $\tilde{\gamma}(0) \neq \tilde{\gamma}(2\pi)$, then γ exhibits monodromy.

(Usually studied for polynomials on points in $\mathbb{C}$.)

We say γ has monodromy of order k if k repetitions of the covering loop returns to the starting point, & k is minimal such value!

$$\tilde{\gamma}(0) = \tilde{\gamma}^k(k2\pi)$$

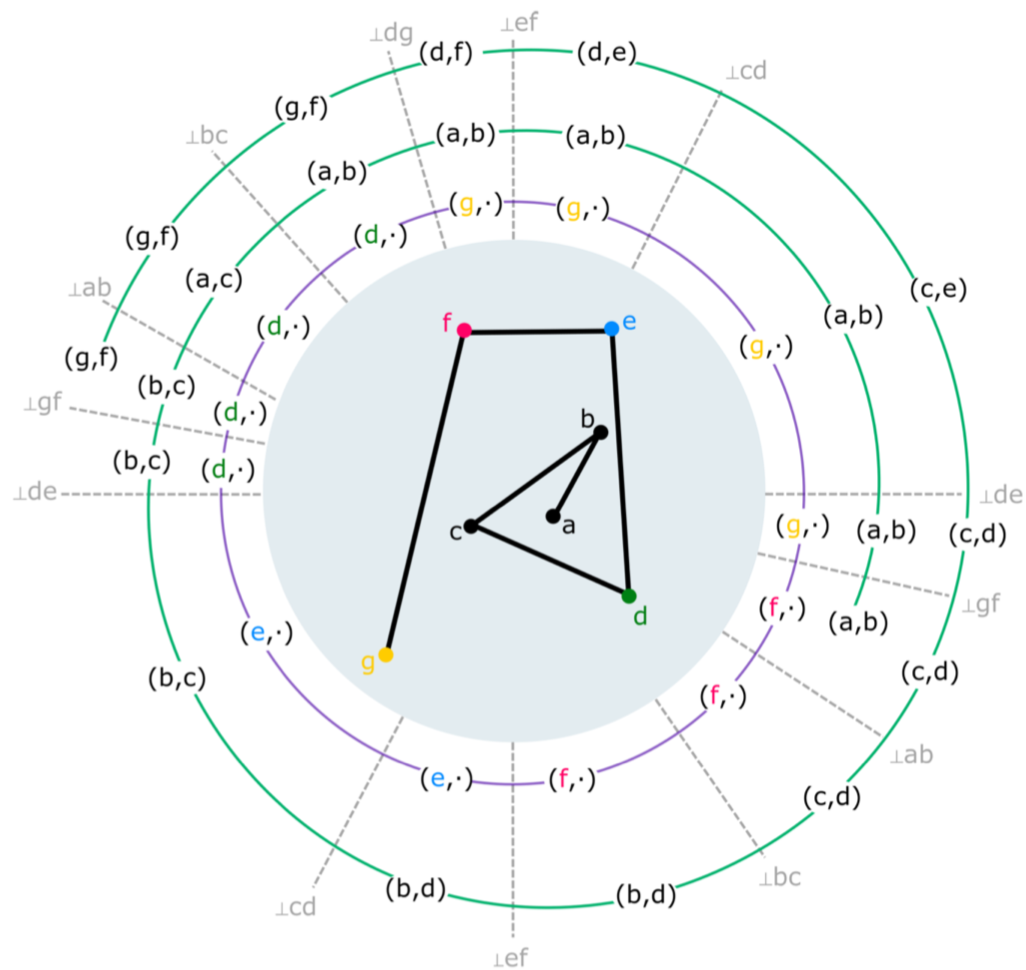
$$= \underbrace{\tilde{\gamma} \circ \cdots \circ \tilde{\gamma}}_{k \text{ times}}$$



Here: $k=2$

Monodromy & directional transforms

Recently Ayra et al 2024 studied monodromy for the directional transform:



(a) PHT for a spiral.

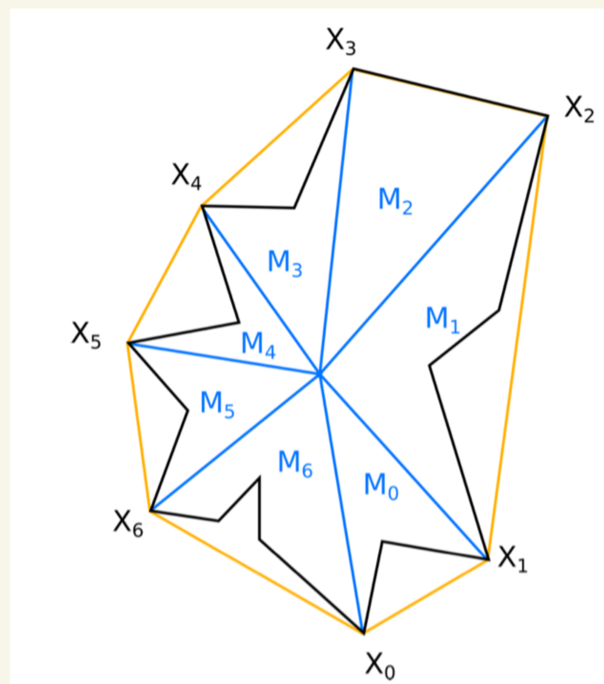
Star-shaped objects in $\mathbb{R}^2$

A shape M is star-shaped if $\exists$ point $c \in M$ such that for any $x \in M$, the line is contained inside of M.

Arya et al prove that, if $A \subseteq \mathbb{R}^2$ & A star-shaped $\Rightarrow$ $PHT(A)$ has no monodromy.

Proof technique: decompose into sectors, & show

$$PH_0(A) \cong \bigoplus_{\text{sectors } M_i} PH_0(M_i)$$



Generalizing

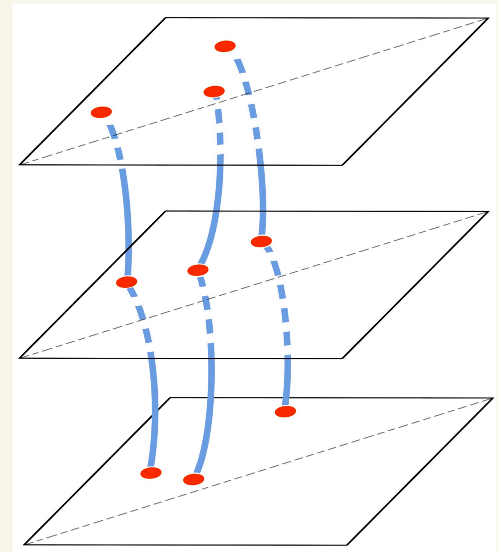
Unfortunately, unclear if reverse holds, so not (yet) a complete characterization of what monodromy is catching.

In $\mathbb{R}^d$ for $d > 3$, proof does not work: there are star-shaped objects where PH_0 cannot be decomposed into sectors.

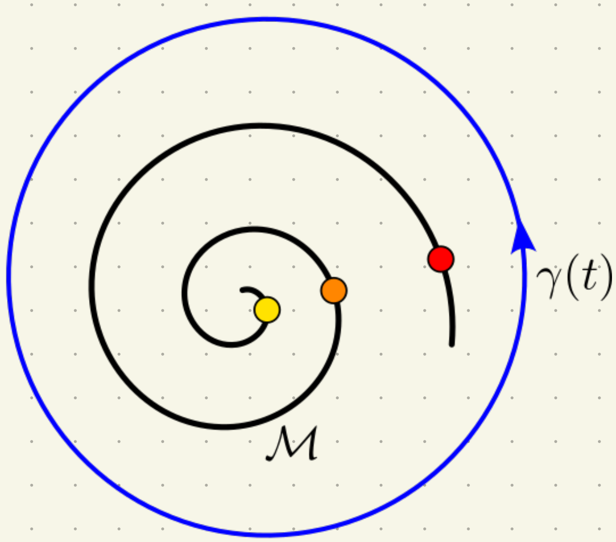
Other direction: how <u>much</u> monodromy can we find?

Vineyard viewpoint:

Monodromy is a permutation of points in "z-axis" →



Back to that spiral:

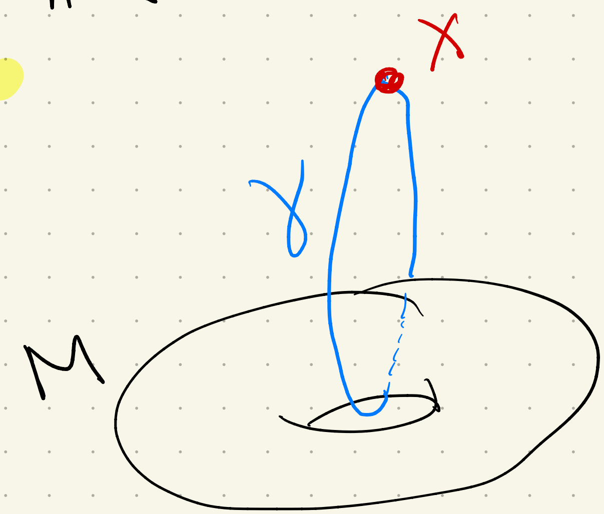


A different transform

$\boxed{\text{Onus et al}}$

Let $d(\cdot, x): M \longrightarrow \mathbb{R}$ be the distance from any $x \in \mathbb{R}^d$ to $M \subset \mathbb{R}^d$

↳ called radial distance fcn



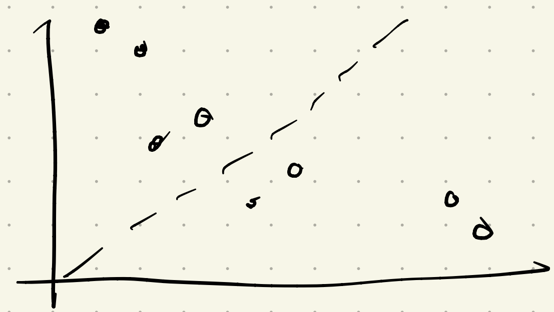
Fix a loop $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^d$

↳ observation loop

Set $x = \gamma(t)$ for $t \in [0, 2\pi]$

↳ family of filtrations $d(\cdot, \gamma(t))_M$

$\mathrm{Dgm}(d(\cdot, x)_M)$



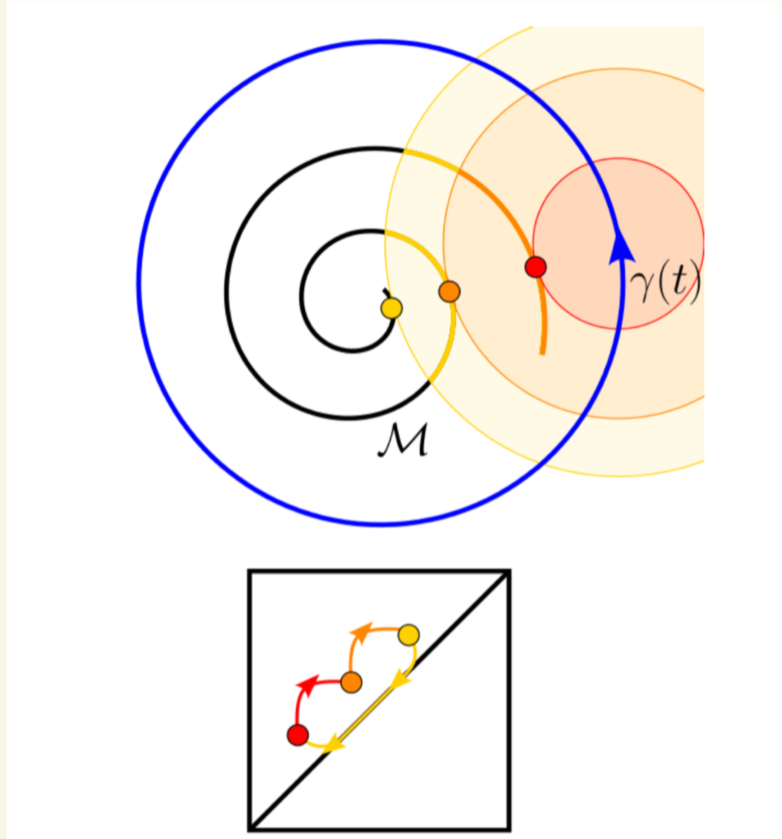
Closed Vineyard map:

$$CV_M : S^1 \longrightarrow S^1 \times \mathrm{Dgm}$$

$$t \longmapsto (t, \mathrm{Dgm}(d(\cdot, \gamma(t))_M))$$

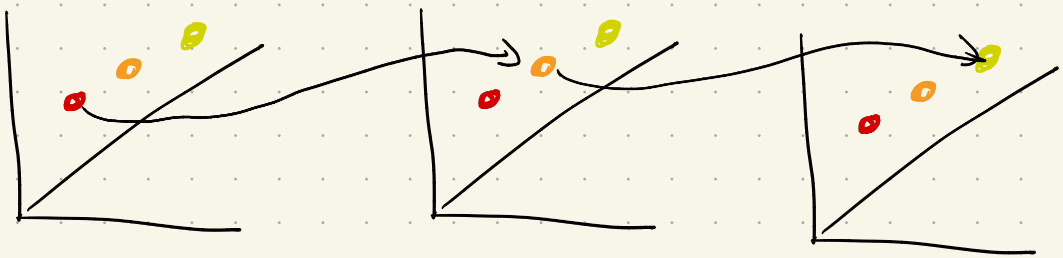
Re-doing the spiral

In this example, the vines will still induce a map from $\mathrm{Dgm}(d(\cdot, \gamma(0))_M)$ to itself which permutes the points



$\mathrm{Dgm}_0(d(\cdot, \gamma(t))_{\mathcal{M}})$

Vineyard:



Here, we have monodromy of order 3.

$\boxed{\text{Arya et al 2024}}$

Can we demonstrate monodromy in higher dimensions?

Which objects have it, & how much?

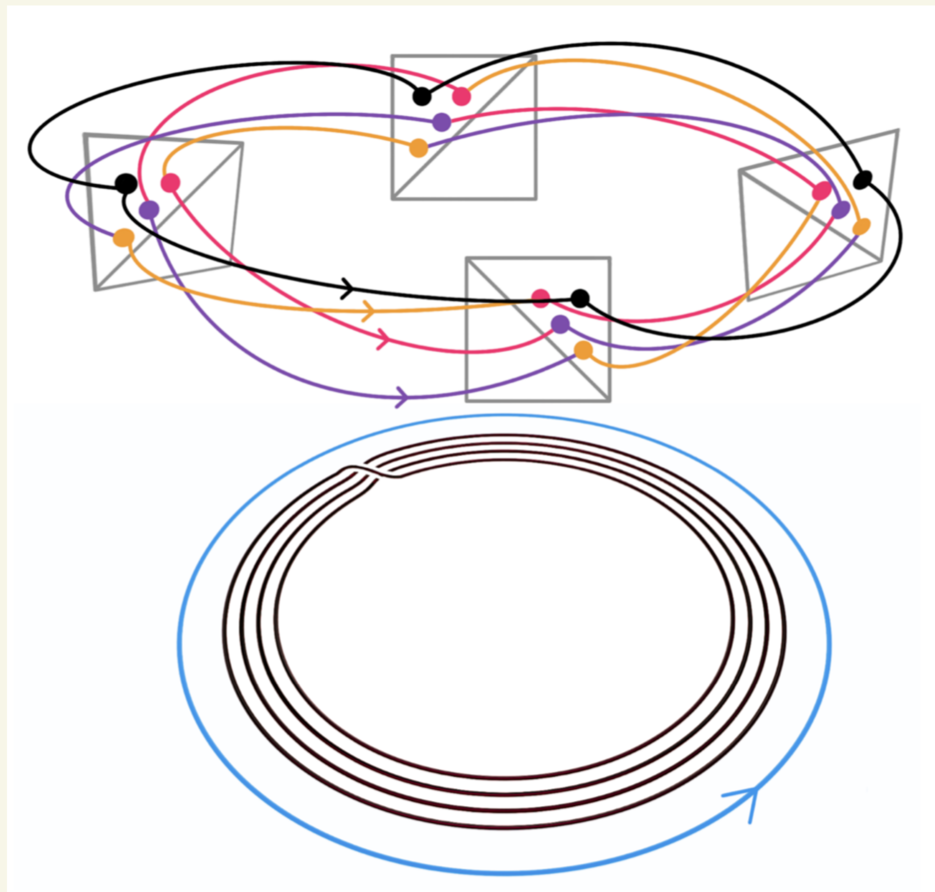
Theorem $\boxed{\text{C., Fillmore, Stephenson, Wintraecken}}$

Monodromy of any order k can be created in the ℓ-vineyard of the radial transform of a manifold M embedded in $\mathbb{R}^{\ell+1}$

Idea for how to prove

We know the radial vineyard can reflects parts of the topology of input space M.

If M were a braid, with period $2\pi k$, we could get order k monodromy.



Braids have previously connected to monodromy:

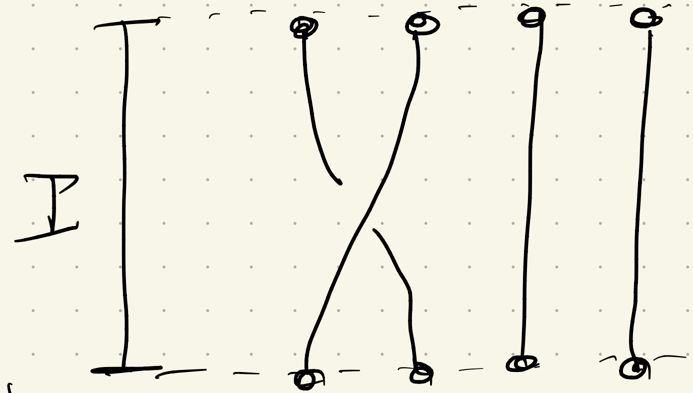
Cohen + Suciu 1997,

Cogolludo-Agustín 2011

Salter 2023, Salter 2024

Braids

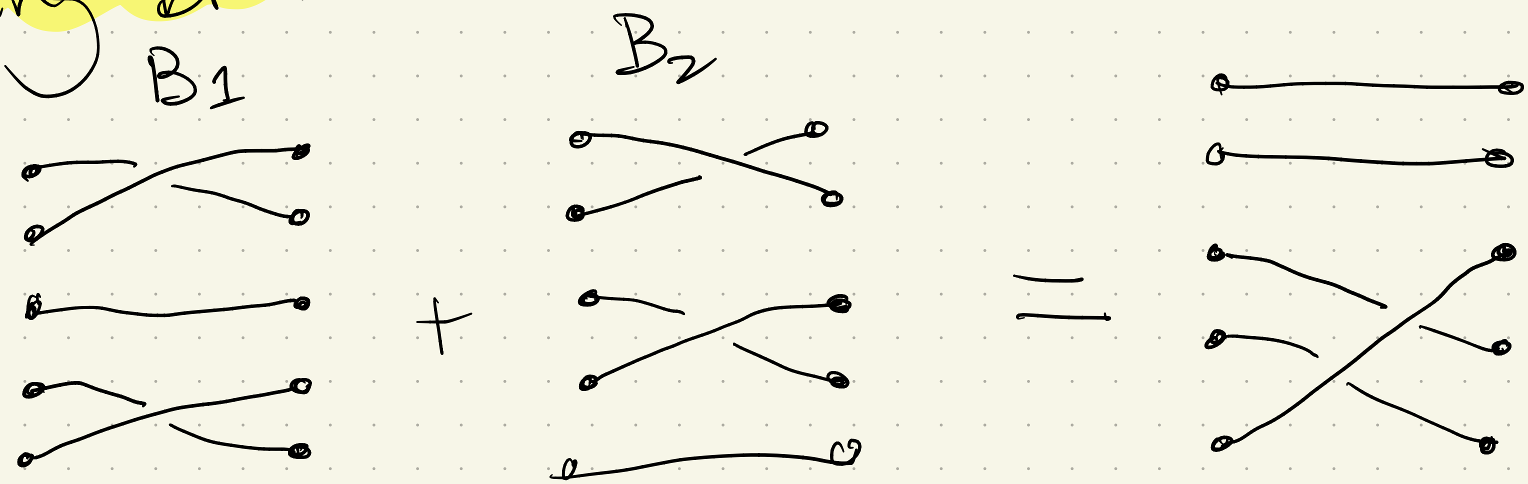
A braid on m strands is the equivalence class of the disjoint union of m intervals $B_i : I \to D^2 \times I$, monotonically inrecsing wrt I, such that end points are a permutation of start points, under ambient braid isotopy.



Variant of Reidemeister moves

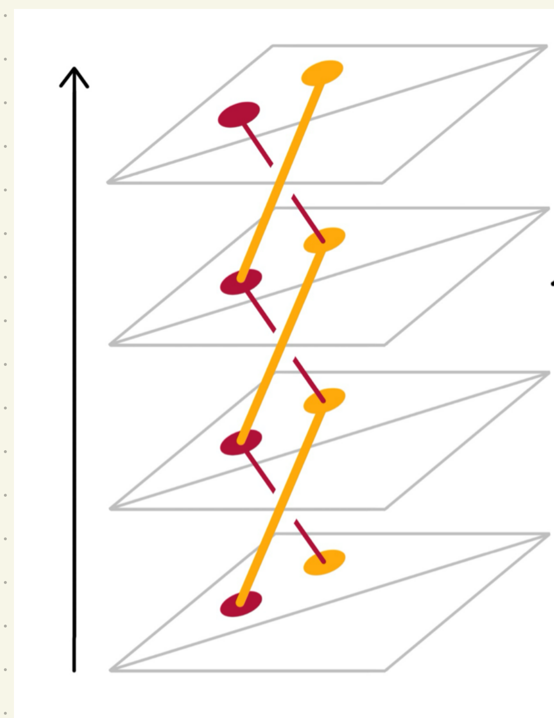
Figure 4: The Reidemeister moves. Left to right: Type I, Type II, Type III.

Composing braids:



If we look a vineyards, can see some of the same patterns

↳ & can ask if they contain braids.

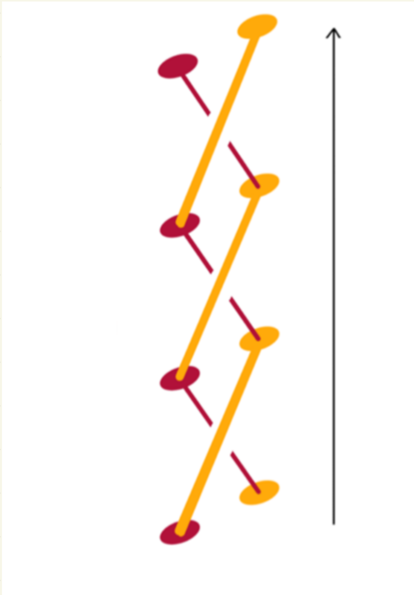


Braids & Knots

If we identify endpoints of a braid & map canonically to torus in $\mathbb{R}^3$, we get a closed braid.

Theorem $\boxed{\text{Alexander 1923}}$

Every knot or link is equivalent to a closed braid.

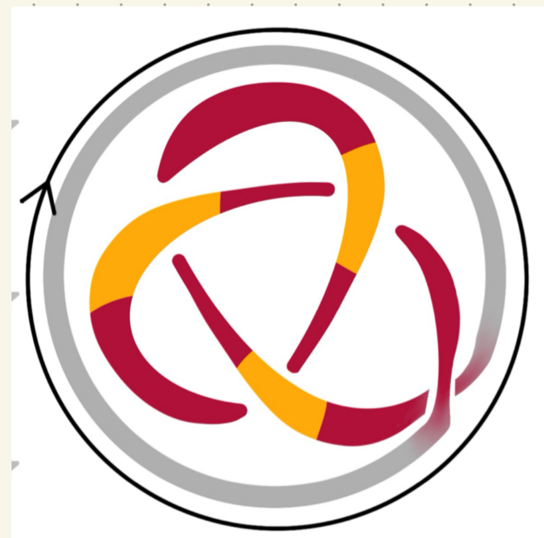


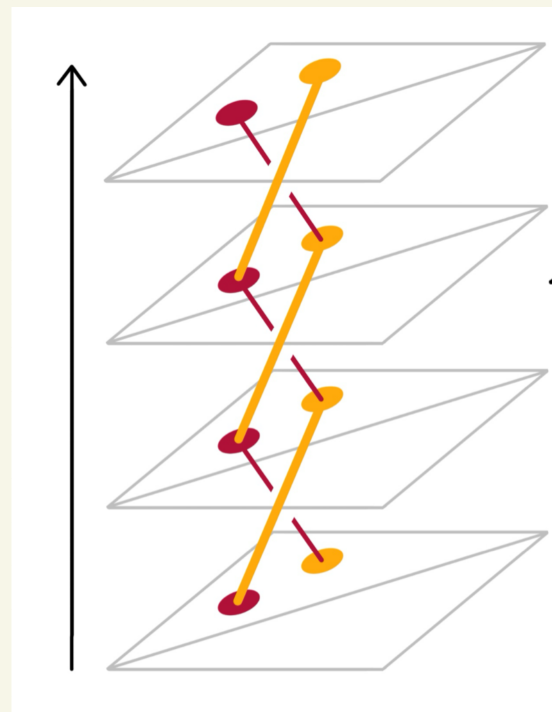
$\Longleftrightarrow$

Question: Can vineyards "braid"?

Theorem $\boxed{\text{C.-Fillmore-Stephenson-Wintraecken}}$

Given any braid B, there exists a manifold $M \subset \mathbb{R}^d$ & a closed curve $\gamma \subset \mathbb{R}^d$ such that identifying the ends of the vineyard of $d(\cdot, \gamma(t))_M$ will yield a braid B' which is equivalent to B after removing spurious components.





Construction

Note that both M and γ must be carefully constructed to work together!

Overview:

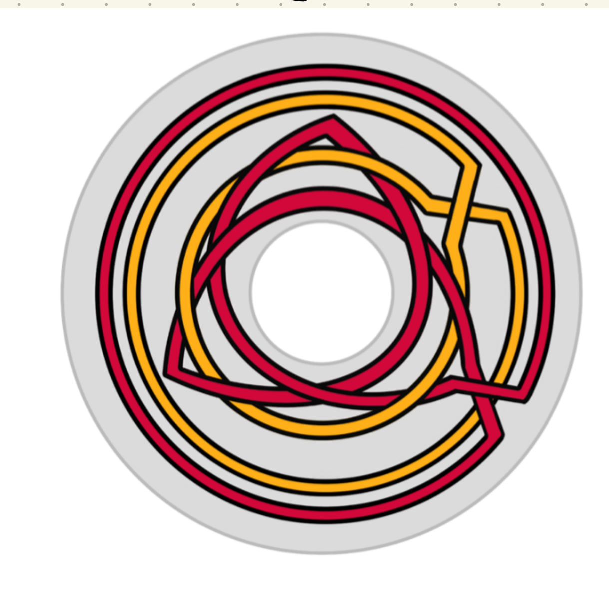
Start with a closed braid $B \subset \mathbb{R}^2$, with k components and s strands.

[Note: Not showing the braid here, but $k=2$ and $s=3$]

We'll convert this to a manifold in $\mathbb{R}^3$

(1) Redraw in a small neighborhood of an annulus, where strands follow fixed radii

⇓



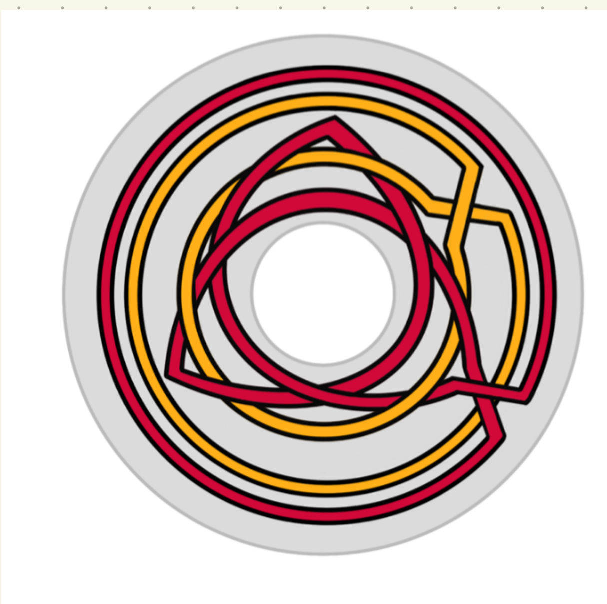
(2) Then, introduce an extra "twist" per component, & wrap loop around outside of annulus

$\Rightarrow$ adds $O(sk)$ crossings and gives $n = s + k$ strands total

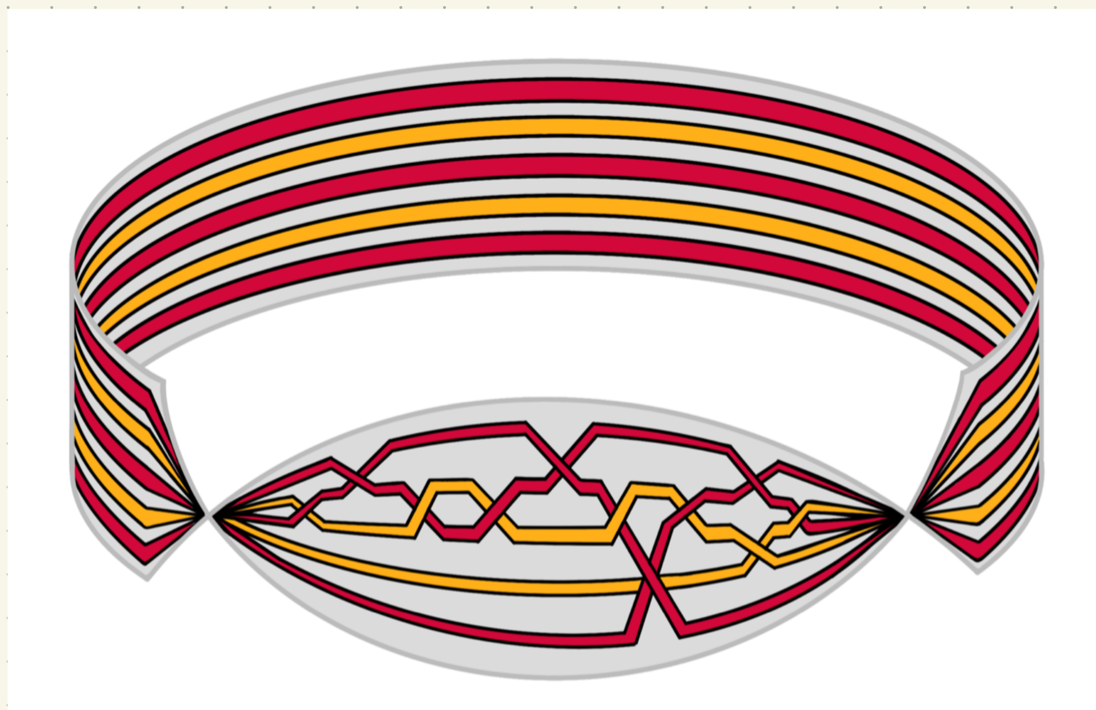
③ Deform B so all crossings occur in fraction of annulus, at regular intervals

④ Twist annulus in 3d, so non-crossing part is orthogonal. (Note: no new crossings!)

⑤ Set observation loop to follow annulus at fixed distance & near "center"



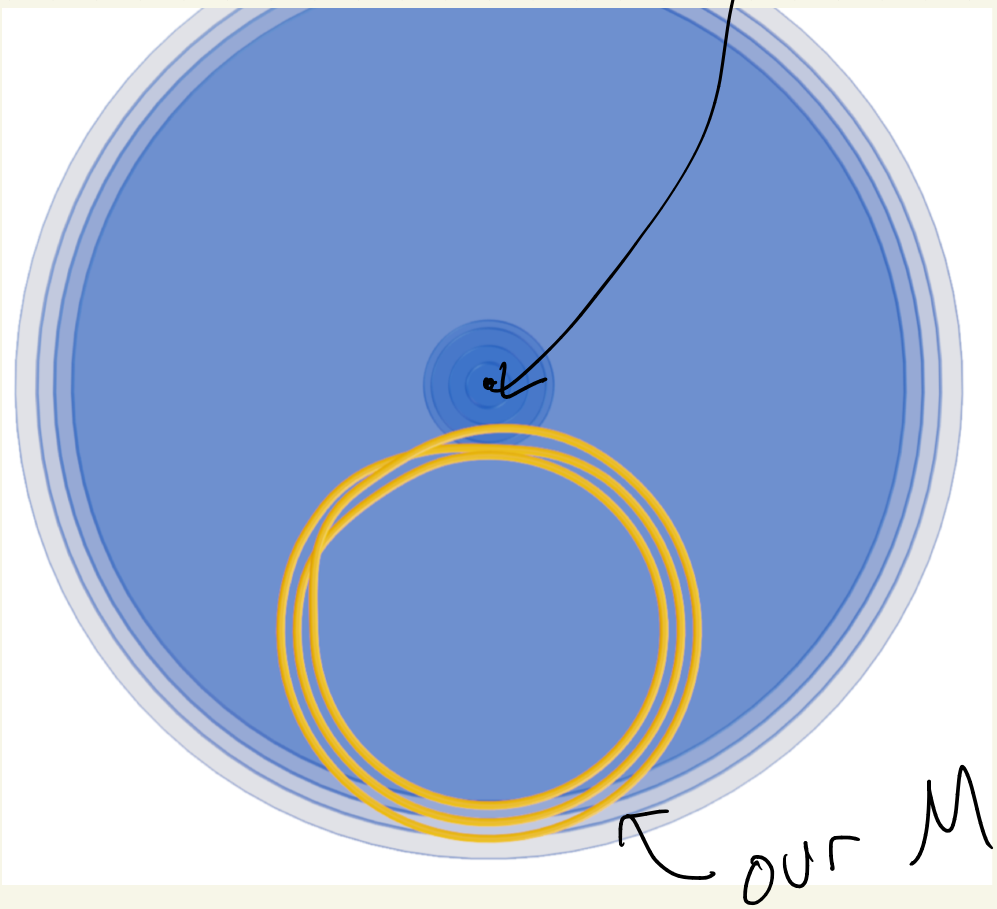
$\Longrightarrow$



The vineyard

Several technical lemmas relying on Morse theory + angles show that births in H_0 all occur before deaths!

This means we can use embedding to control vineyard diagrams..

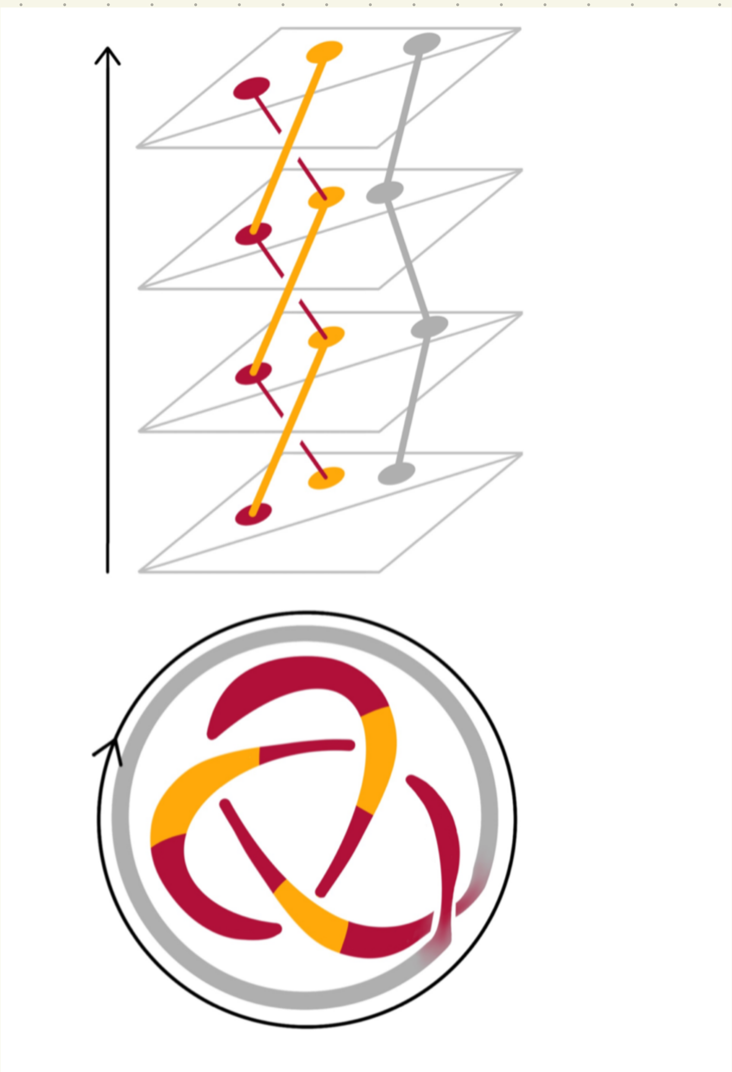


Back to that extra loop

Elder rule in persistence
$\Rightarrow$ first birth + last
death are paired.

We added an outer strand
to account for this.

Result: For each component,
will be an unlinked
strand, which will be
an extra circle in
vineyard.

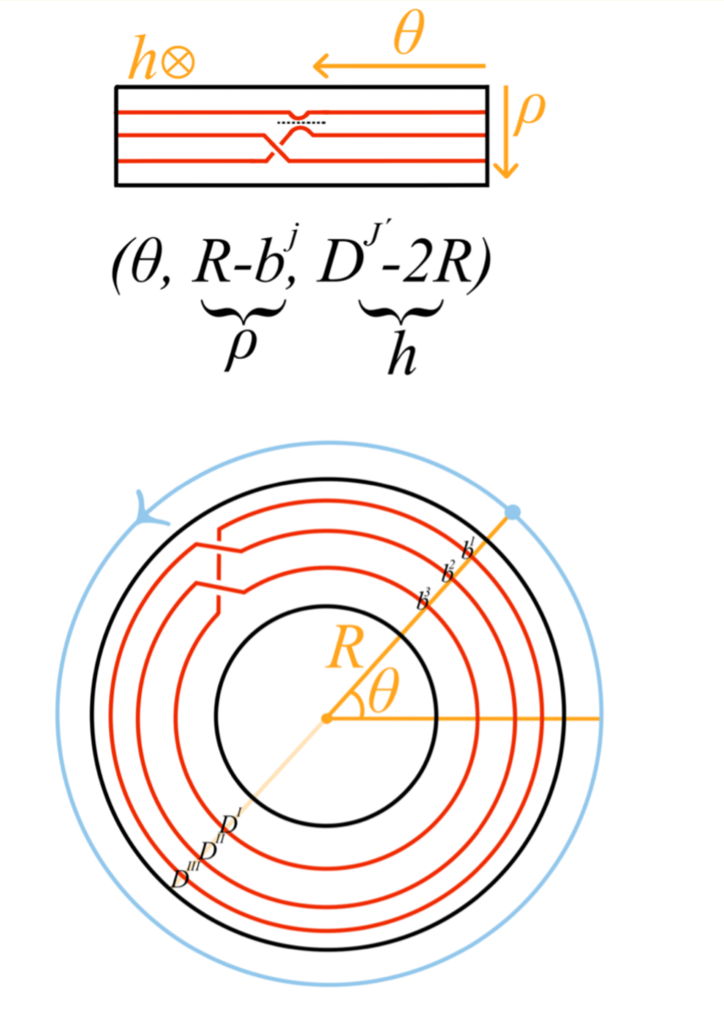


Parameterizing the vineyard

Our vines have annular coordinates: θ, h & ρ

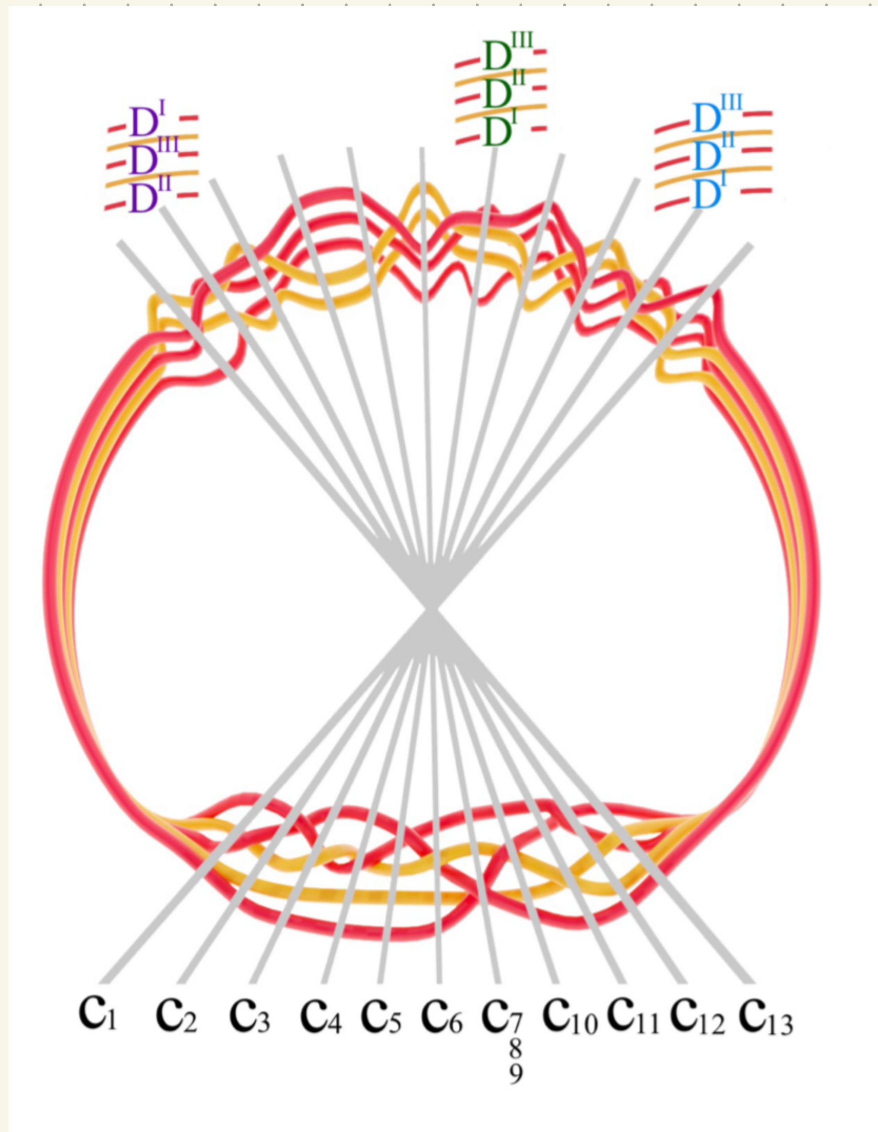
Away from crossings, vines have distinct ρ so h doesn't matter.

To get correct over/under crossings in vineyard, we need to play with the geometry of our embedding a bit...



Pushing or Pulling

Our construction ensured that crossings are evenly spaced, & all deaths occur at opposite points.
Therefore, we perturb deaths to make sure vineyard has correct crossings.



(Note: hiding some intense calculations here...)

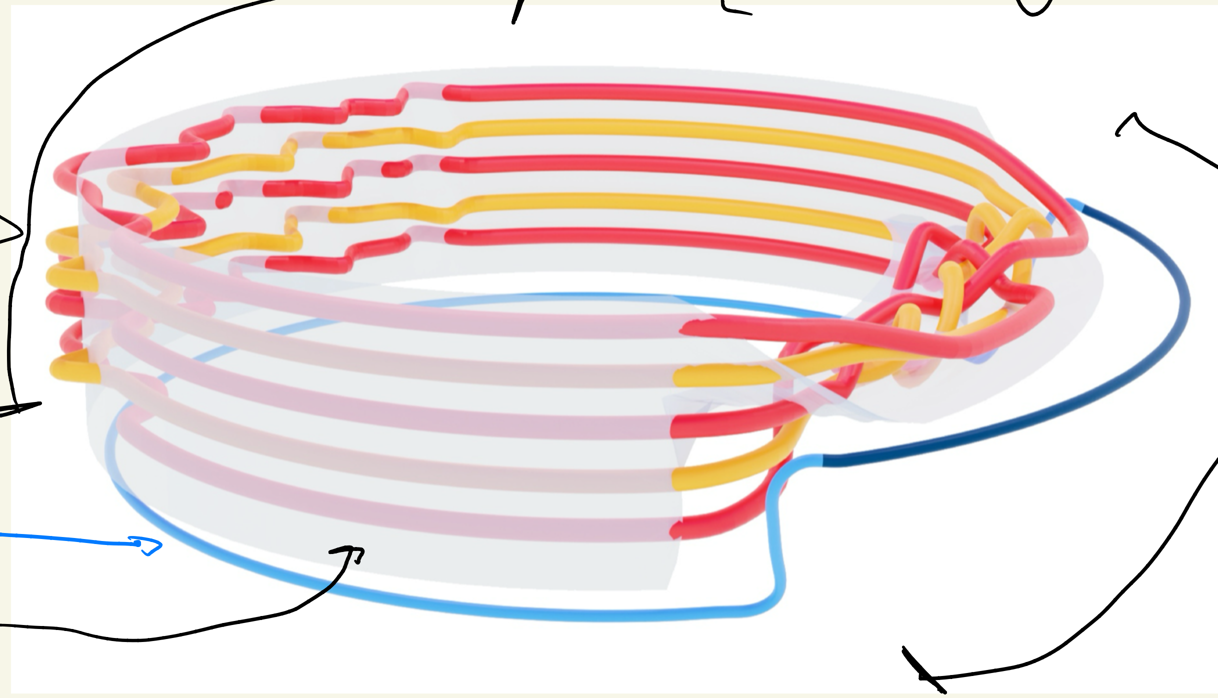
Final result: braids

all {crosings}

Pushed & pulled deaths

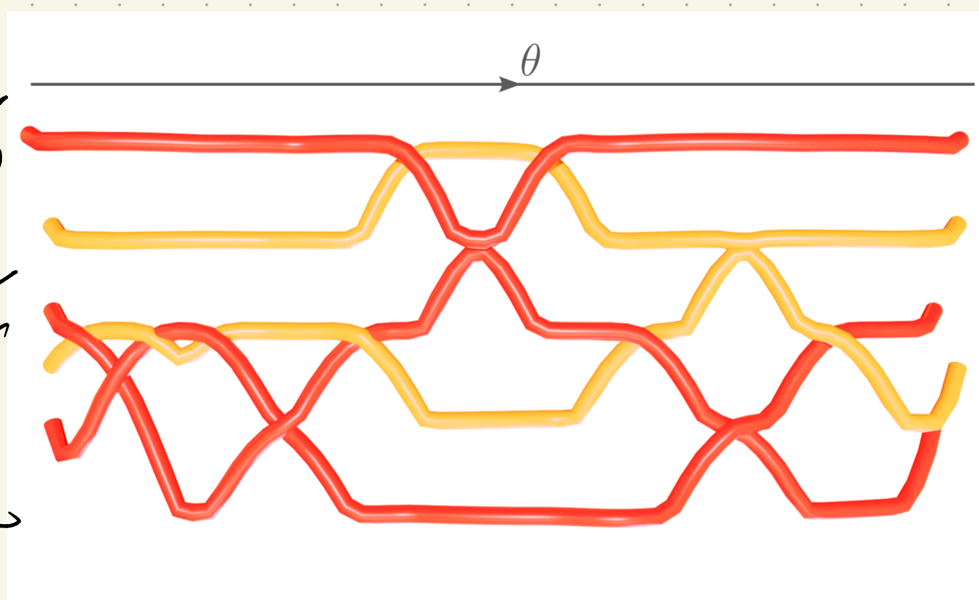
observation loop

base annulus



unlinked outer loops

braid



Vineyard of dark blue portion of the observation loop

Higher dimensions

To have this calculation work for higher dimensions, take $(\ell+1)$-dimensional α-offset of braid $B \times O$ in $\mathbb{R}^3 \oplus \mathbb{R}^{\ell-1} < \mathbb{R}^d$.

↳ Given this manifold construction ($\&$ same base loop), vines in our ℓ-vineyard will be α-close to the O-vineyard.

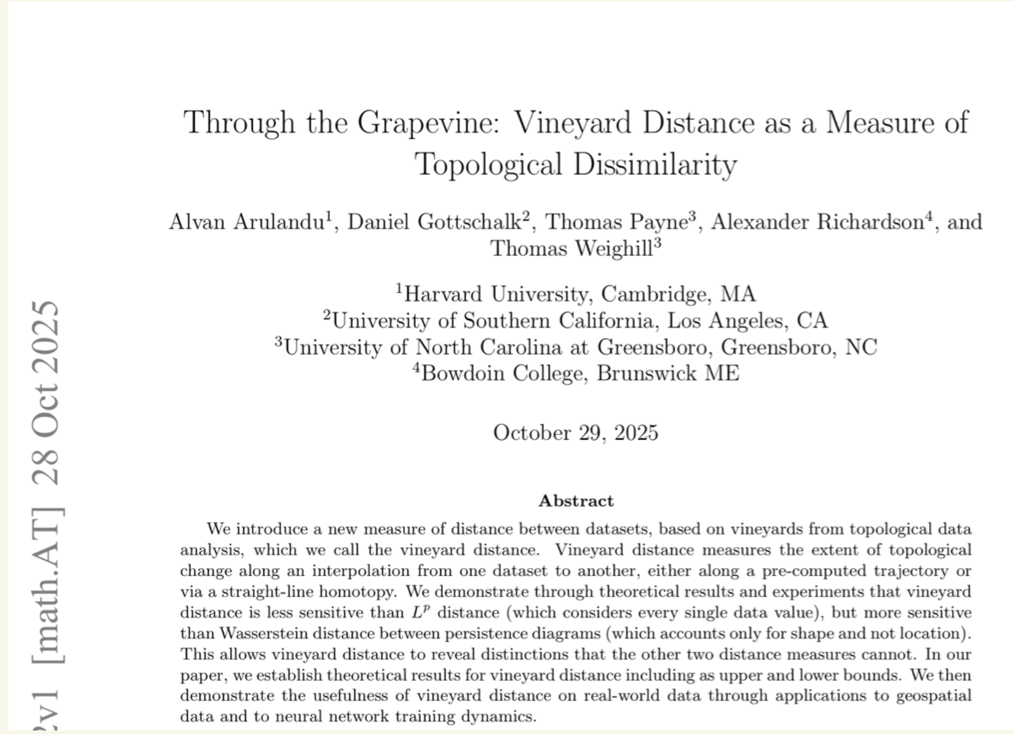
So why study vineyards?

Recognizing knots is not easy

↳ but classifying vineyards might involve knots

That said, classifying knots is very well studied!

Other (new!) approaches ignore the knots:



2v1 [math.AT] 28 Oct 2025

Through the Grapevine: Vineyard Distance as a Measure of Topological Dissimilarity

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Abstract

We introduce a new measure of distance between datasets, based on vineyards from topological data analysis, which we call the vineyard distance. Vineyard distance measures the extent of topological change along an interpolation from one dataset to another, either along a pre-computed trajectory or via a straight-line homotopy. We demonstrate through theoretical results and experiments that vineyard distance is less sensitive than L^p distance (which considers every single data value), but more sensitive than Wasserstein distance between persistence diagrams (which accounts only for shape and not location). This allows vineyard distance to reveal distinctions that the other two distance measures cannot. In our paper, we establish theoretical results for vineyard distance including as upper and lower bounds. We then demonstrate the usefulness of vineyard distance on real-world data through applications to geospatial data and to neural network training dynamics.

- Decompose into vines
- Project vines into x-y plane
- Measure length there