TDA - Fall 2025

Maps

Morse theory

Recap

- Overview of class
  - ↳ questions?
- At some point, check HW page for overview of the class project
- Office hours: Monday after class
  Tues or Thurs.

Question from last time

Metric space:

a pair $(\mathbb{T}, d)$, where $\mathbb{T}$ is a set and $d: \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfies

other: $d(p,q) \geq 0$

- $d(p,q) = 0 \iff p = q$
- $d(p,q) = d(q,p) \quad \forall p,q \in \mathbb{T}$
- triangle inequality: $d(p,q) \leq d(p,r) + d(r,q) \quad \forall p,q,r \in \mathbb{T}$

Sometimes a 4th: $\forall p,q \quad d(p,q) \geq 0$

But: the first 3 imply the 4th.

Why? (exercise)

A topological space is disconnected if $\exists$ 2 disjoint nonempty open sets $U, V \in \mathcal{T}$ s.t. $\mathbb{T} = U \cup V$.

(The space is connected if it is not disconnected.)

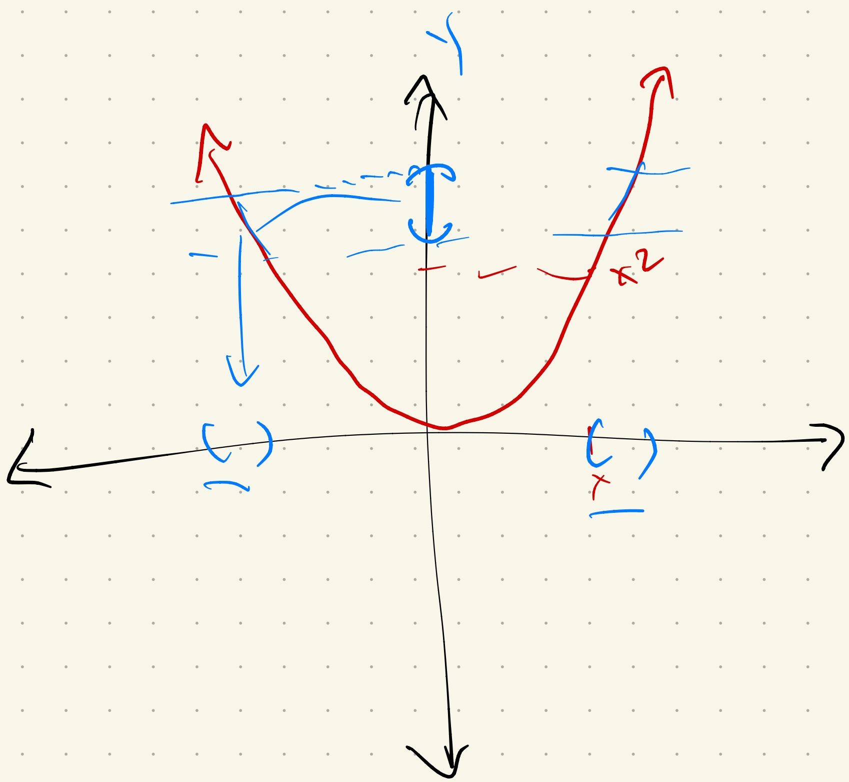
Ex: $A = (1,2) \cup (3,4) \subset \mathbb{R}$

Note: subspace topology: Given $\mathbb{U} \subseteq \mathbb{T}$, $\mathbb{U}$ can inherit topology from $\mathcal{T}$ via
$$\{X \cap U \mid X \in \mathcal{T}\}$$

Maps

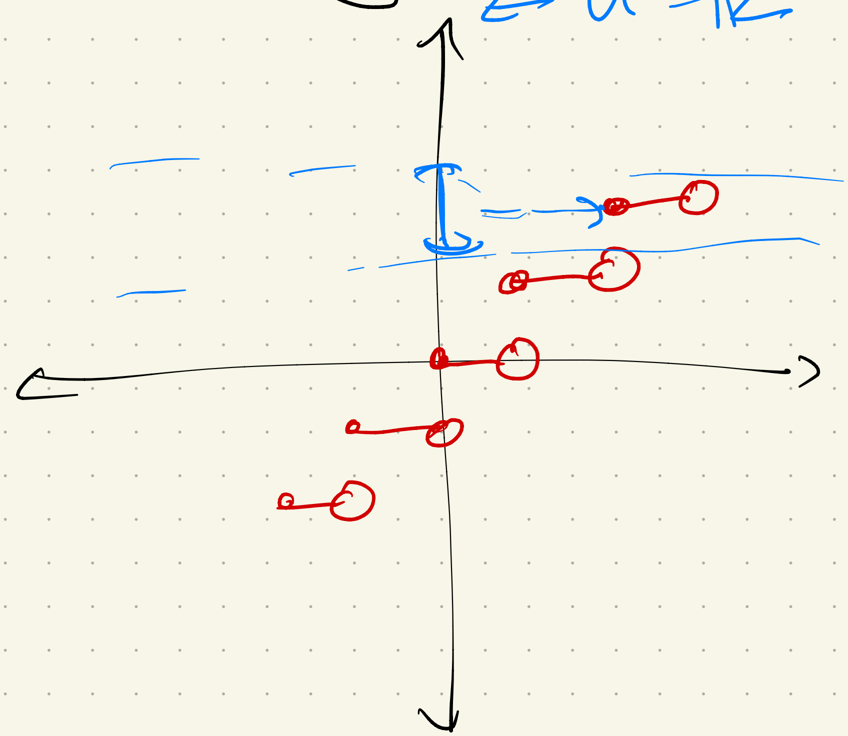
A function $f: T \to U$ is **continuous** if for every open set $Q \subseteq U$, $f^{-1}(Q)$ is open.
(These are also called **maps**.)

Example: $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$



Example: $g: \mathbb{R} \to \mathbb{R}$
$g(x) = \lfloor x \rfloor$

$\leftarrow U = \mathbb{R}$

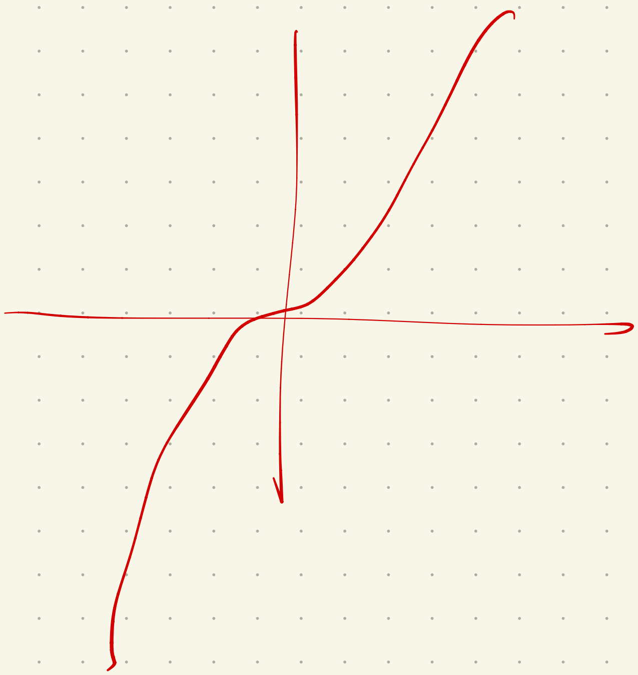


A map $f: T \to U$ is an **embedding** of T into U if f is injective.

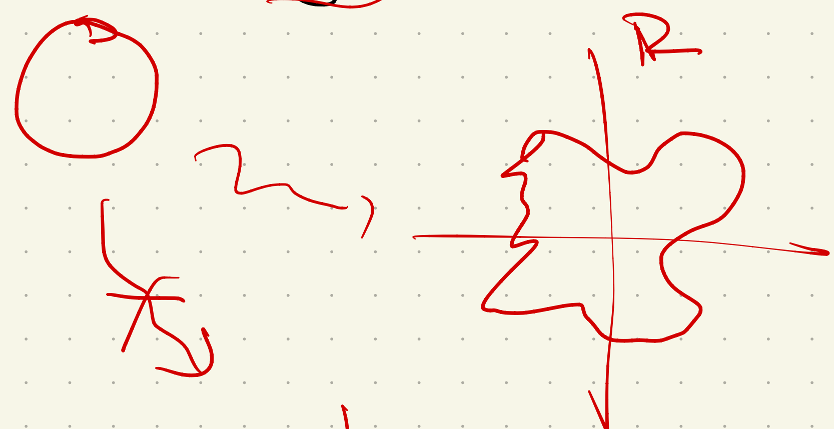
$\boxed{\text{injective, or 1-1: } f(a)=f(b) \Leftrightarrow a=b}$

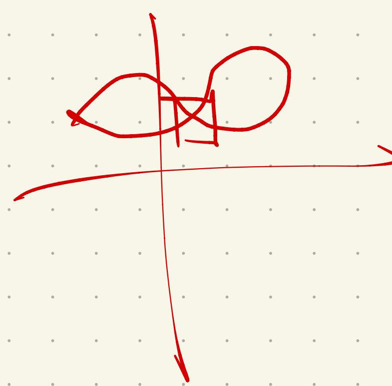
Example: $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = x^3$$



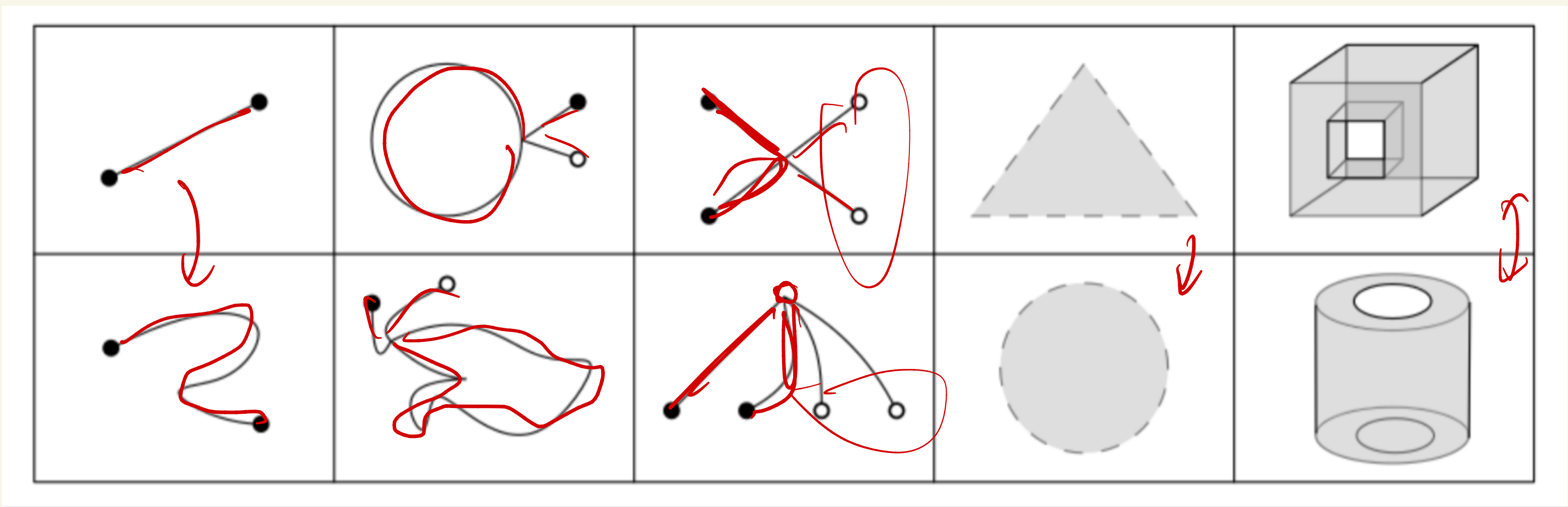
Example: $g: S^1 \to \mathbb{R}^2$





Let T & U be topological spaces.
A homeomorphism $h: T \to U$ is a bijective map whose inverse is also continuous.
(We say T & U are homeomorphic if such an h exists.)

Examples:



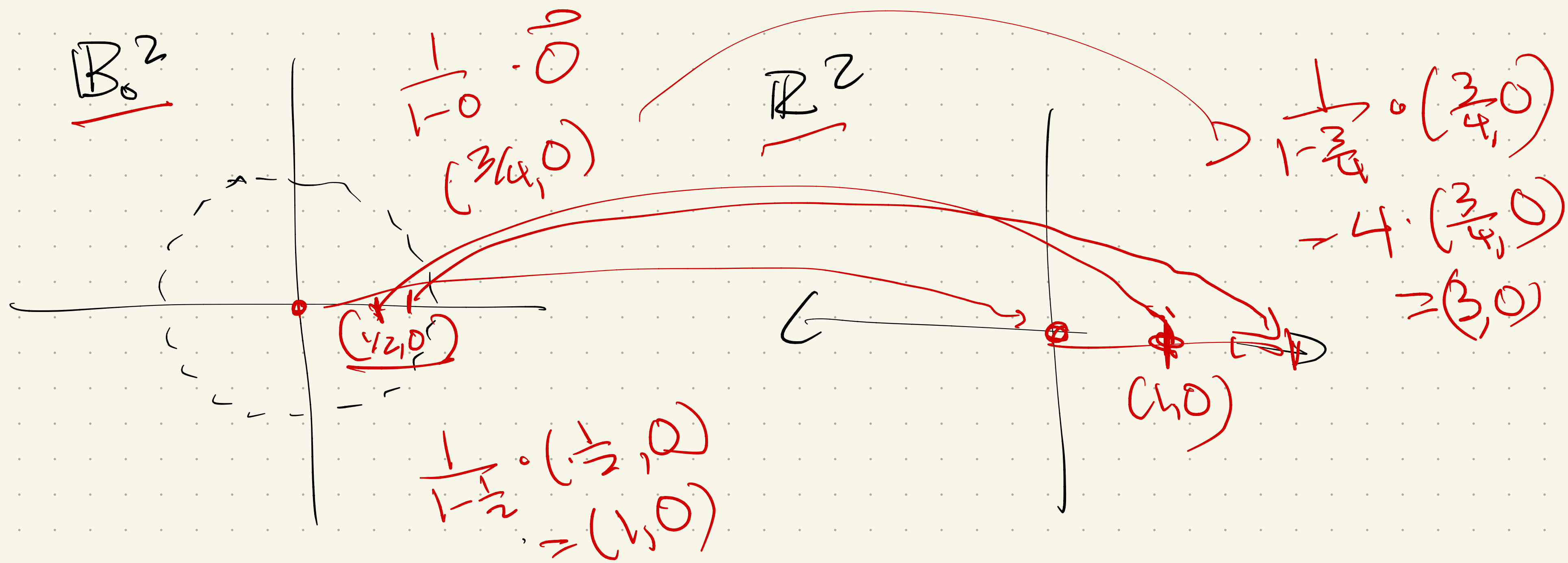
Note: requires construction of a function!

Example: open d-ball $\mathbb{B}_0^d$ and $\mathbb{R}^d$

$$h(x) = \frac{1}{1-\|x\|}\cdot x$$

(so $h^{-1}(y) = \dfrac{\sqrt{1+4\|y\|}-1}{2\|y\|^2}\cdot \vec{y}$ if $\vec{y} \neq 0$, &

$= 0$ if $y = 0$)



For nice enough spaces, a "cheap trick":

Proposition

If T & U are compact metric spaces, every bijective map $T \to U$ has a continuous inverse.

Isotopy

When $\mathbb{T}, \mathbb{U}$ are subspaces of a common topological space, can study something stronger:

An isotopy connecting $\mathbb{T} \subseteq \mathbb{R}^d$ & $\mathbb{U} \subseteq \mathbb{R}^d$ is a map $\xi : \mathbb{T} \times [0,1] \to \mathbb{R}^d$ where

- $\xi(\mathbb{T}, 0) = \mathbb{T}$
- $\xi(\mathbb{T}, 1) = \mathbb{U}$
- $\forall\, t \in [0,1]$, $\xi(\cdot, t)$ is a homeomorphism from $\mathbb{T}$ to its image

Ambient isotopy: map $\xi : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$

Examples: For open d-ball again:

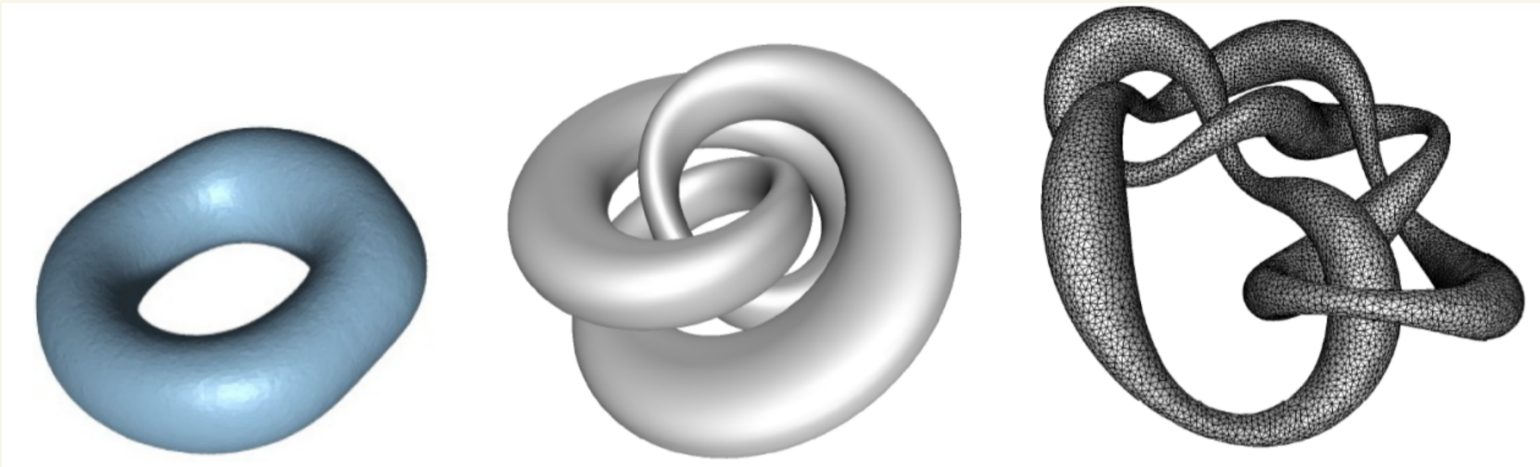
Consider $\xi(x,t) = \dfrac{1-(1-t)\|x\|}{1-\|x\|} \cdot \vec{x}$

if $t=0$: $\dfrac{1-\cancel{(1-0)}^{1}\|x\|}{1-\|x\|} \cdot \vec{x} = \vec{x}$

if $t=1$: $\dfrac{1-(1-1)\cdot\|x\|}{1-\|x\|} \cdot \vec{x} = \dfrac{1}{1-\|x\|} \cdot \vec{x}$

So $\mathbb{B}_0^d$ & $\mathbb{R}^d$ are isotopic.

Homeomorphism << isotopy << ambient isotopy :



Obstruction comes from the ambient space : $\mathbb{R}^3 \setminus$ knot here

Homotopy

Consider maps $g: X \to U$ and $h: X \to U$.

A homotopy is a map $H: X \times [0,1] \to U$ such that $H(\cdot,0) = g$ and $H(\cdot,1) = h$

$g: B_0^3 \hookrightarrow \mathbb{R}^3$

Example:

$g: B_0^3 \to \mathbb{R}^3$ inclusion map $h(\vec{x}) = \vec{x}$

$h: B_0^3 \to \mathbb{R}^3$, $h(\vec{x}) = \vec{0}$ $\leftarrow$

homotopy: $H(x,t) = (1-t)\cdot\vec{x}$

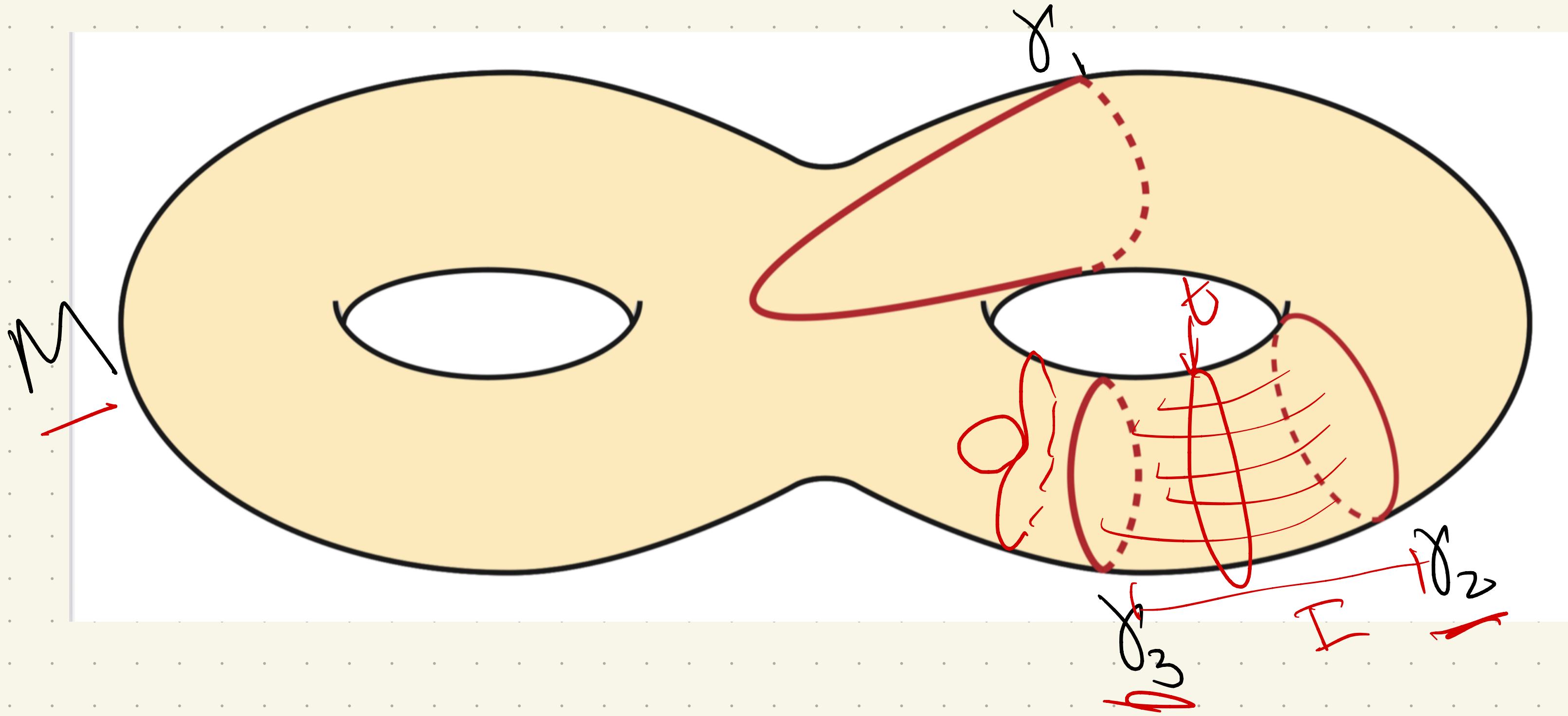
check: $H(\cdot,0) = (1-0)\cdot\vec{x} = \vec{x}$

$H(\cdot,1) = (1-1)\cdot\vec{x} = 0$

& between:

Another: curves on surfaces

$$\gamma_1, \gamma_2, \gamma_3 : S^1 \longrightarrow M$$



Here, homotopy $H : \overline{S^1 \times [0,1]} \longrightarrow M$

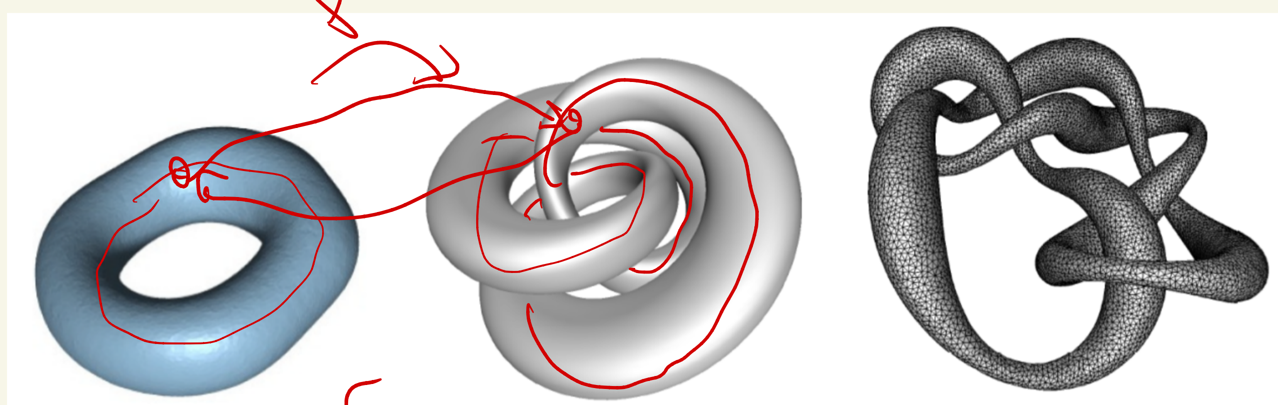
$H(\cdot, 0) = \gamma_3 \qquad H(\cdot, 1) = \gamma_2$

Homotopy equivalence

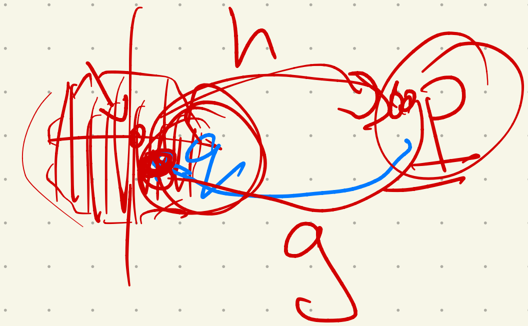
Two topological spaces $\mathbb{T}$ + $\mathbb{U}$ are homotopy equivalent if $\exists\, g: \mathbb{T} \to \mathbb{U}$ and $h: \mathbb{U} \to \mathbb{T}$ such that $h \circ g$ and $g \circ h$ are homotopic to identity maps

Example:

g



h



Another: $\mathbb{B}_0^2$ and any point p.

$h : \mathbb{B}_0^2 \longrightarrow \{p\}$, $h(x) = p$

$g : \{p\} \to \mathbb{B}_0^2$, with $g(p) = q$

(an arbitrary point in $\mathbb{B}_0^2$)

$h \circ g : \{p\} \longrightarrow \{p\}$ ✓

$g \circ h : \mathbb{B}_0^2 \longrightarrow \mathbb{B}_0^2$

sends every $x \in \mathbb{B}_0^2$ to q

Homotopy: $H(x, t) = (1-t) \cdot q + t \cdot x$

at $t=0$: q

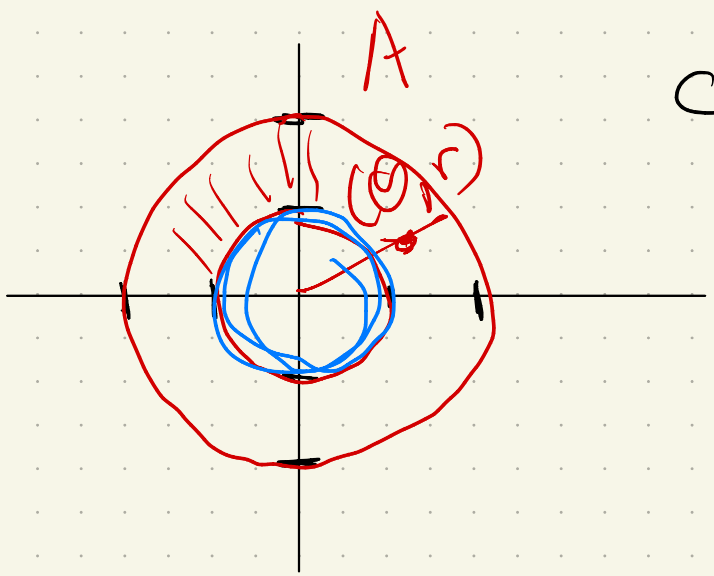
at $t=1$: $t \cdot \bar{x} = \bar{x}$

Retracts

Consider $\mathbb{T}$ a topological space, & $\mathbb{U} \subseteq \mathbb{T}$ a subspace.

A retraction r of $\mathbb{T}$ to $\mathbb{U}$ is a map $f: \underline{\mathbb{T}} \to \underline{\mathbb{U}}$ s.t. $\underline{f(x) = x} \quad \forall x \in \mathbb{U}$.

Example: annulus $A = \{ \underline{(\theta, r)} \mid 1 \leq r \leq 2$ and $\theta \in [0, 2\pi] \}$



circle $S^1 = \{ (\theta, 1) \mid \theta \in [0, 2\pi] \}$

How to make f?

$f(\theta, r) = (\theta, 1)$

Deformation Retract

$\mathbb{U} \subseteq \mathbb{T}$ is a deformation retract if the identity map on $\mathbb{T}$ can be continuously deformed to a retraction with no change to points in $\mathbb{U}$. More precisely:

$\exists$ homotopy $R : \mathbb{T} \times [0,1] \to \mathbb{T}$ s.t.

- $R(\cdot, 0) = \mathrm{id}_{\mathbb{T}}$
- $R(\cdot, 1)$ is a retraction $\mathbb{T} \to \mathbb{U}$
- $R(x,t) = x$ for every $x \in \mathbb{U}$ and $t \in [0,1]$.

Try previous example:

annulus $A = \{(\theta, r) \mid 1 \le r \le 2 \text{ and } \theta \in [0, 2\pi]\}$

circle $S^1 = \{(\theta, 1) \mid \theta \in [0, 2\pi]\}$

Set $R((\theta, r), t) = (\theta, (1-t)\cdot r + t)$

Check 3 things:

if $t=0$: $(\theta, r+t)$

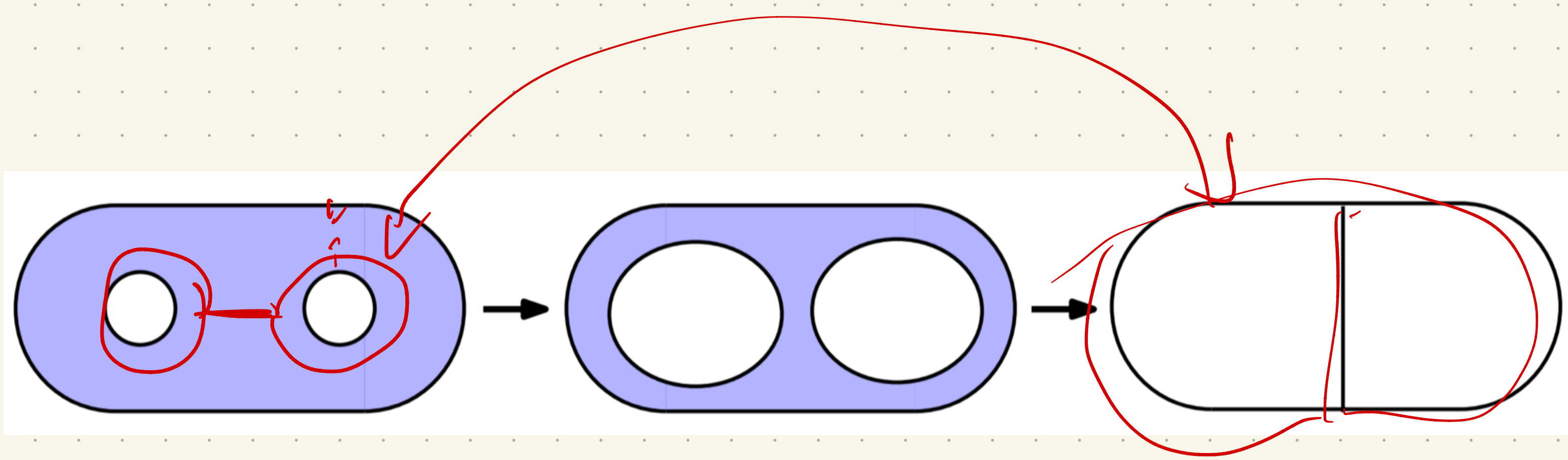
if $t = 1$ (θ, t)

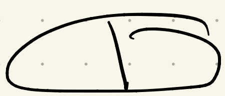
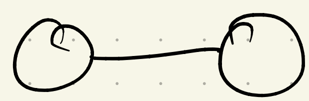
& $R((\theta, 1), t) = (\theta, (1-t)\cdot 1 + t)$
$= (\theta, 1)$

Why care??

Theorem If U is a deformation retract of T, then T & U are homotopy equivalent.

Ex:



(Note:  and  are homotopy equivalent, but no deformation retract.)

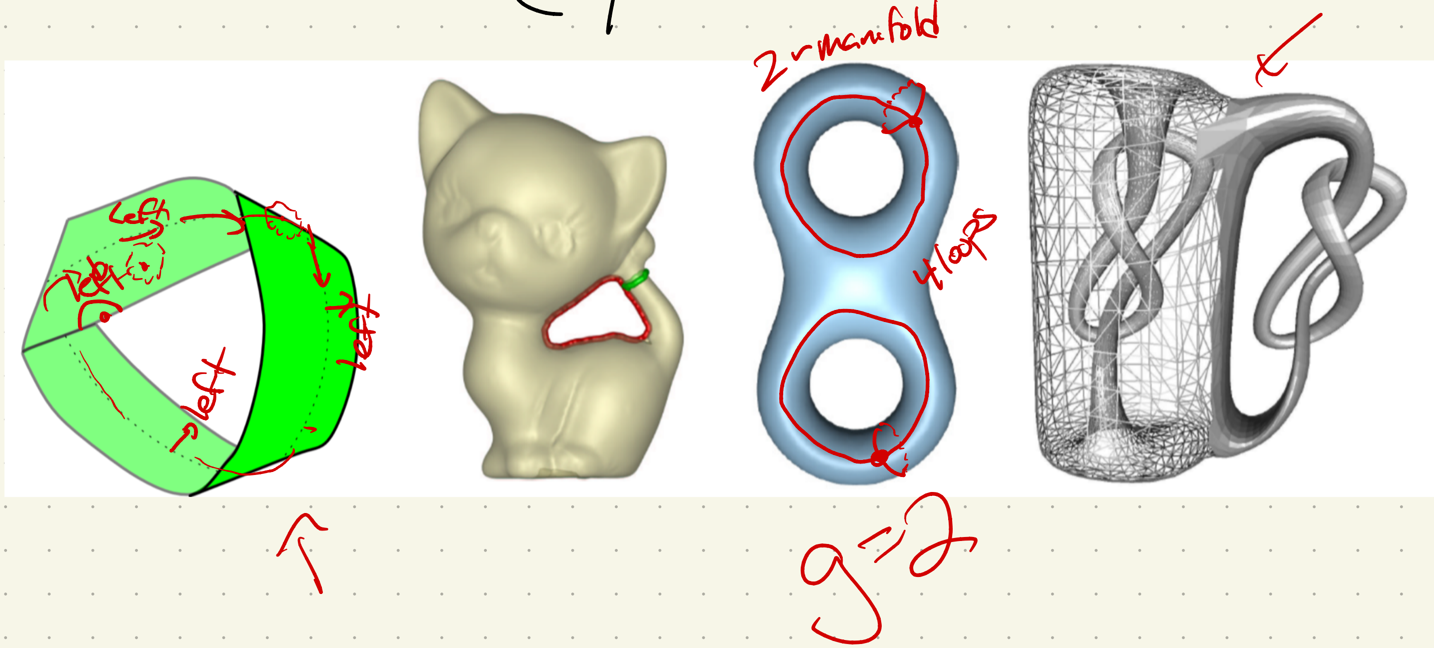
Manifolds

A topological space is an m-manifold if every $x \in M$ has a point homeomorphic to the m-ball $\mathbb{B}_o^m$ or the m-halfspace $\mathbb{H}^m$:

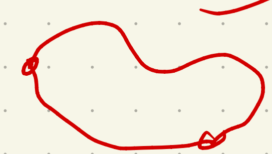
$\mathbb{B}_o^2$

$$\mathbb{B}_o^m = \{ y \in \mathbb{R}^m \mid \|y\| < 1 \}$$

$$\mathbb{H}^m = \{ y \in \mathbb{R}^m \mid \|y\| < 1 \text{ and } y_m \geq 0 \}$$



Notation / terminology

- boundary : look like $\mathbb{H}^d$
- Surface : 2-manifold
- Non-orientable : walk along a curve starting on one side. If you could end up on other side when you return $\rightarrow$ non-orientable
- Loop : 1-manifold, no boundary $\mathbb{R}$ 
- Genus g : $\exists$ a set of $2g$ loops which can be removed without disconnecting it.

Smooth

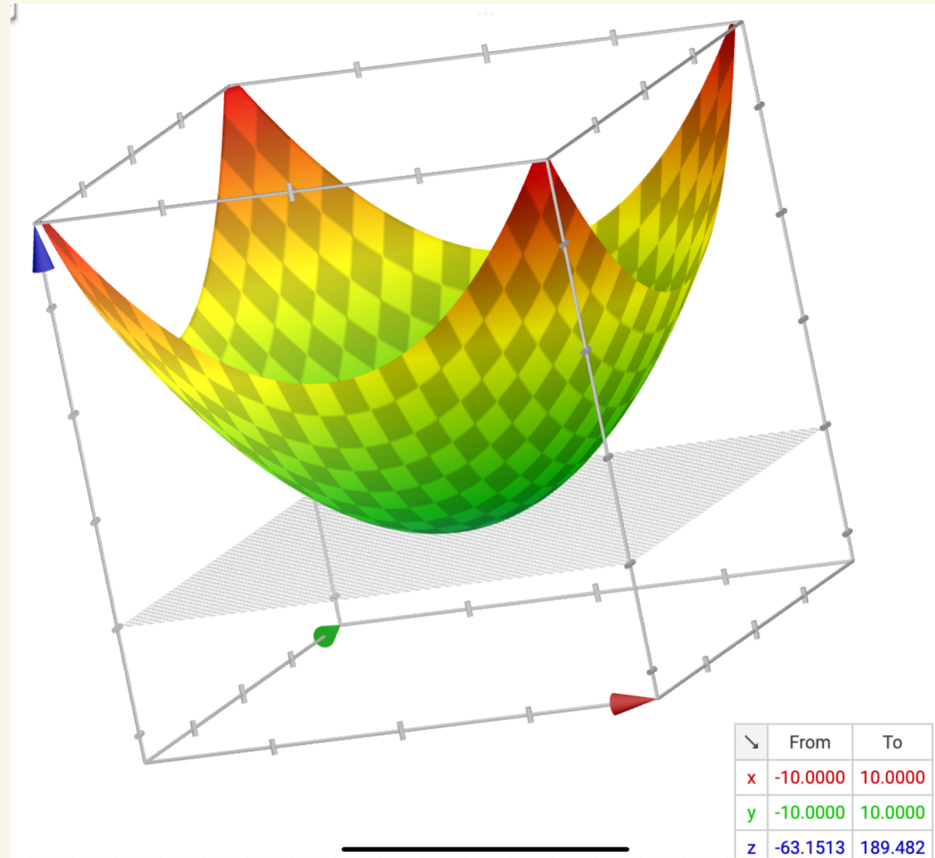
Topological manifolds are spaces
But usually, consider an embedding
into Euclidean space $\Rightarrow$ geometry.

Given a smooth function $f: \mathbb{R}^d \to \mathbb{R}$,
the gradient vector field $\nabla f: \mathbb{R}^d \to \mathbb{R}^d$
at a point x is:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right]$$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f((x_1, x_2)) = x_1^2 + x_2^2$$



$$\nabla f = \left[\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f\right]$$

$$= [2x_1, 2x_2]$$

Then $\nabla f((0,0)) = [0,0]$

$\nabla f((1,0)) = [2,1]$

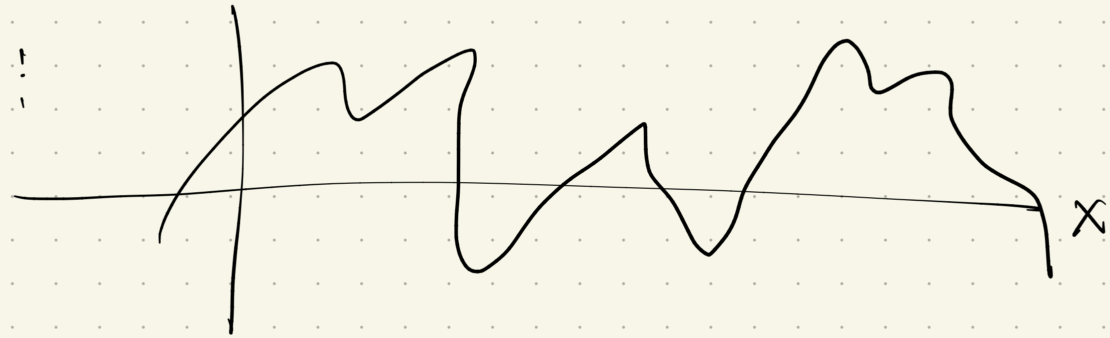
Critical point

Any $p \in \mathbb{R}^d$ where $\nabla f(p) = \vec{0}$

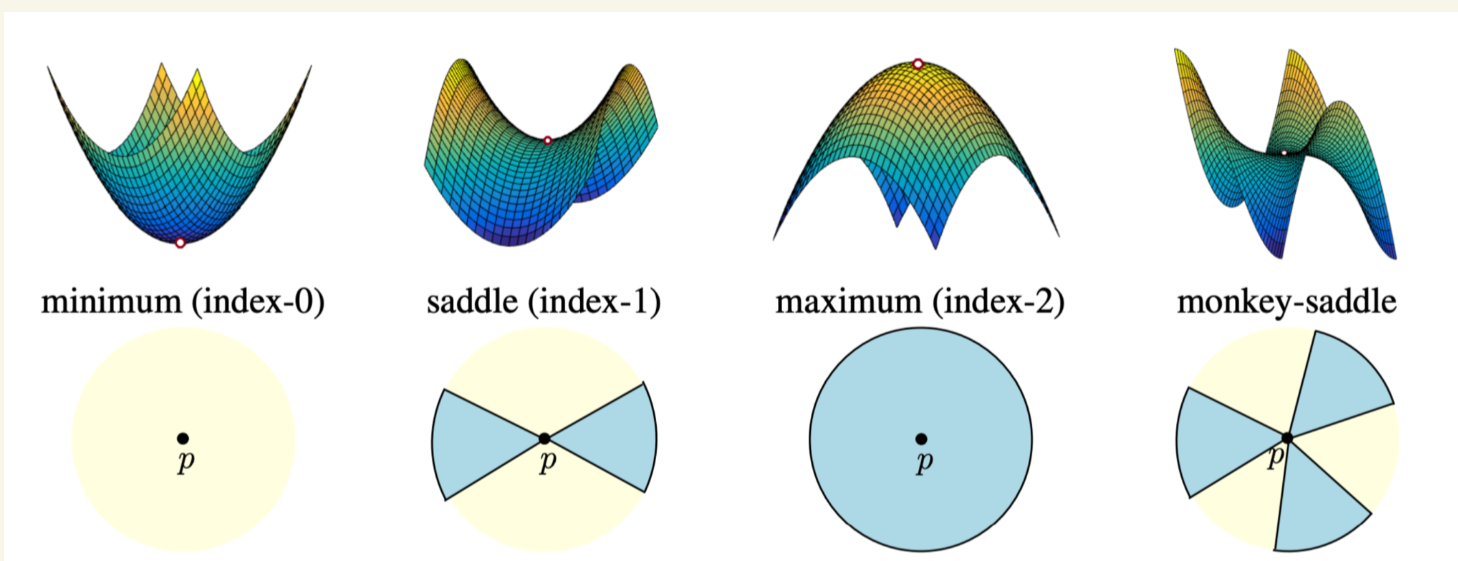
(Otherwise we say p is regular)

On 1 manifolds:

$$\frac{\partial f}{\partial x} x = 0$$



On 2 manifolds.



Extending to manifolds:

Given $\Phi : U \to W$, $U \subseteq \mathbb{R}^k$ & $W \subseteq \mathbb{R}^d$ open sets, where

$$\Phi(x) = (\phi_1(x), \ldots, \phi_d(x))$$

The Jacobian of Φ is a $d \times k$ matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial \phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \phi_1(x)}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_d(x)}{\partial x_1} & \cdots & \frac{\partial \phi_d(x)}{\partial x_k} \end{bmatrix}$$

Types of critical points

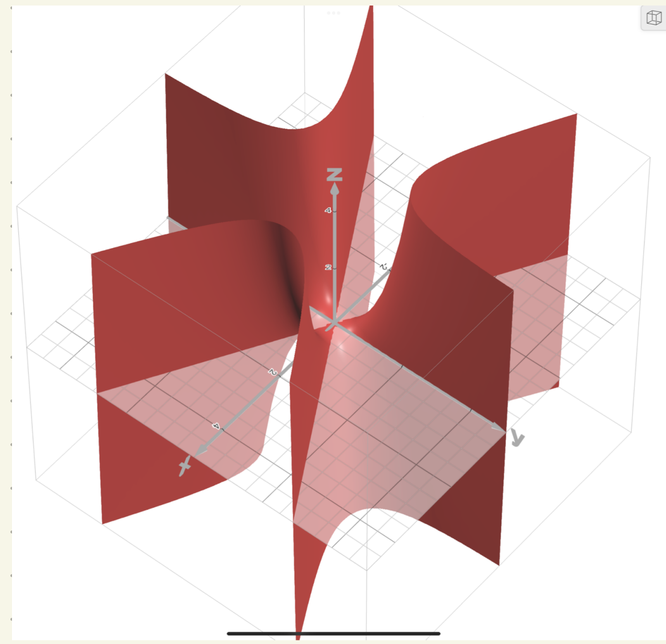
For a smooth m-manifold, the Hessian matrix of $f: M \to \mathbb{R}$ is the matrix of 2nd order partial derivatives!

$$Hessian(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix}$$

A critical point is non-degenerate if Hessian is nonsingular ($\det \neq 0$); otherwise degenerate.

An example: $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^3 - 3x_1 x_2^2$$



$\nabla f =$

$\Rightarrow$

Is it degenerate?

Hessian: $\begin{pmatrix} \frac{\partial}{\partial x_1 \partial x_1} & \frac{\partial}{\partial x_1 \partial x_2} \\ \frac{\partial}{\partial x_2 \partial x_1} & \frac{\partial}{\partial x_2 \partial x_2} \end{pmatrix} =$

So at $(0,0)$, $\det =$

Morse Lemma

Given a smooth function $f: M \to \mathbb{R}$ defined on a smooth manifold M, let p be a non-degenerate critical point of f. Then $\exists$ a local coordinate system in a neighborhood $U(p)$ s.t.

- p's coordinancte is $\vec{0}$
- locally, any x is in the form

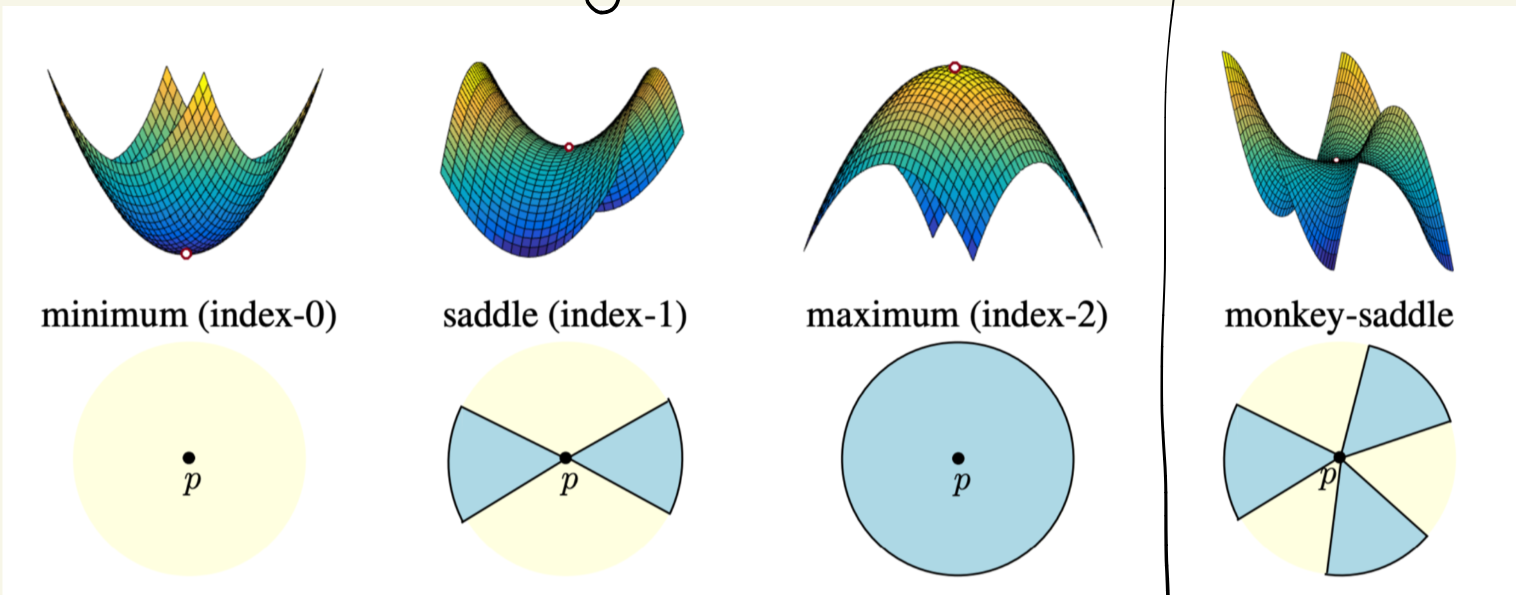
$$f(x) = f(p) - x_1^2 - \cdots - x_s^2 + x_{s+1}^2 + \cdots + x_m^2$$

for some $s \in [0, m]$

s is called the index of p.

Back to that picture...

non-degenerate | degenerate



everything is bigger around p

one coordinate bigger, one smaller

everything is smaller around p

Next time:

Why we care about Morse theory...