TDA - fall 2025

Morse-Smale Complexes

Recap

- No Thurs. office hours this week

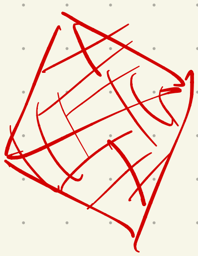
- Welcome back!

- Next assignment: project proposals
  - ↳ submit individually by Nov. 5
  - ↳ 2-3 pages: proposed topic, brief survey of known results, potential plan of attack + some half-baked ideas

- Projects will be in groups of 1-3 students
  - ↳ presentation (~15 minutes) last week or so of class
  - ↳ 10-15 page document by Dec. 14

A discrete Morse function f on a complex K is a function $f: K \to \mathbb{R}$ s.t. for every p-simplex $\sigma \in K$,

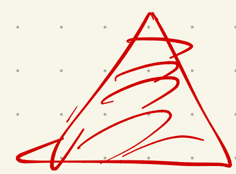
σ:



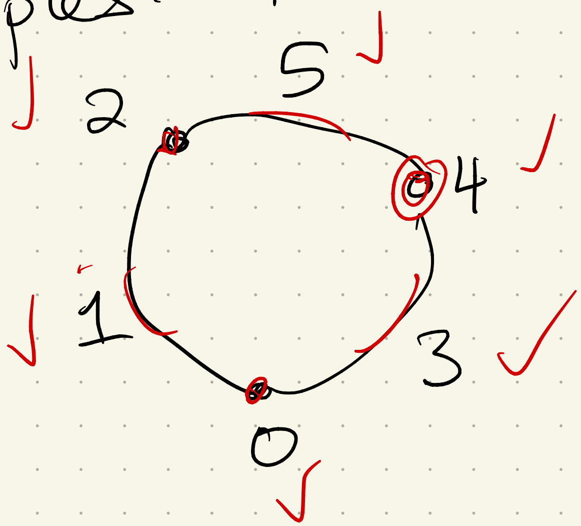
$$\left|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}\right| \leq 1$$

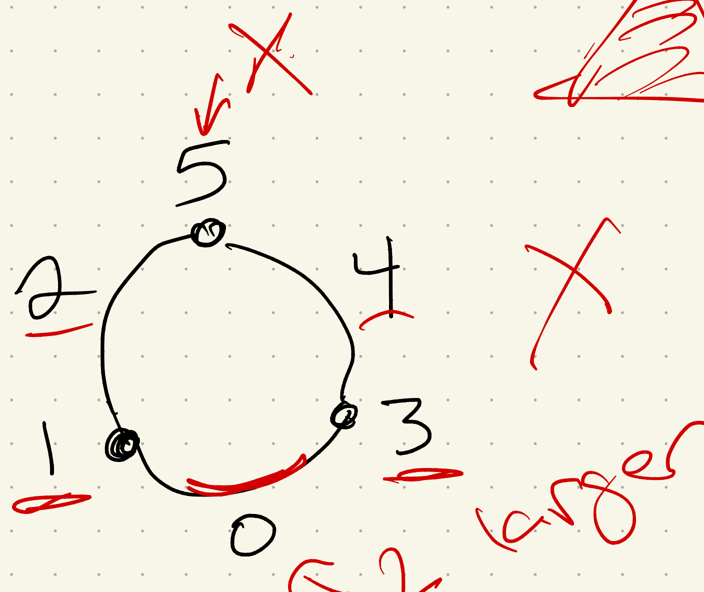
and

$$\left|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}\right| \leq 1$$



Examples: Yes/no?



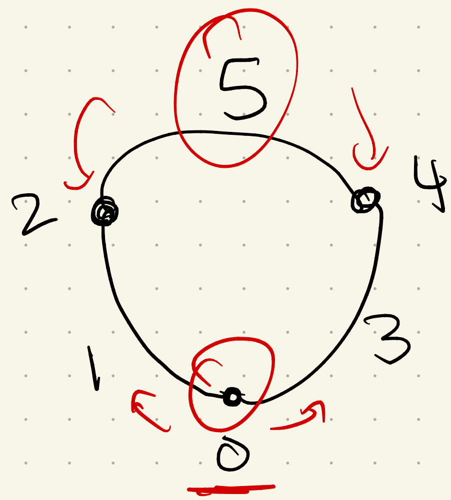


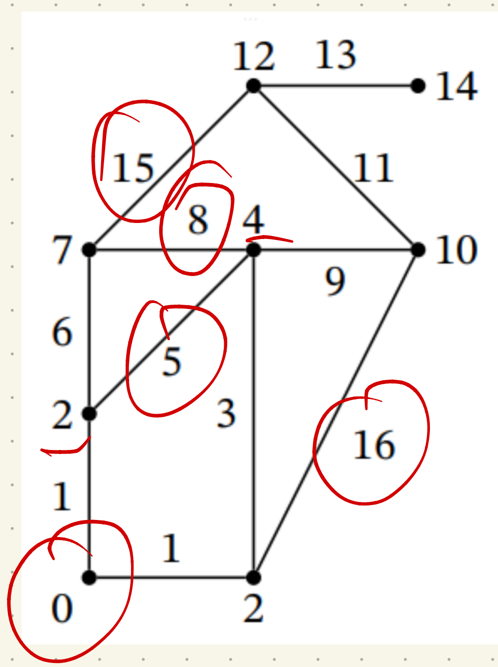
Critical Simplices

A p-simplex is critical with respect to f if $\left|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}\right| = 0$

<u>and</u> $\left|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}\right| = 0$

Example:



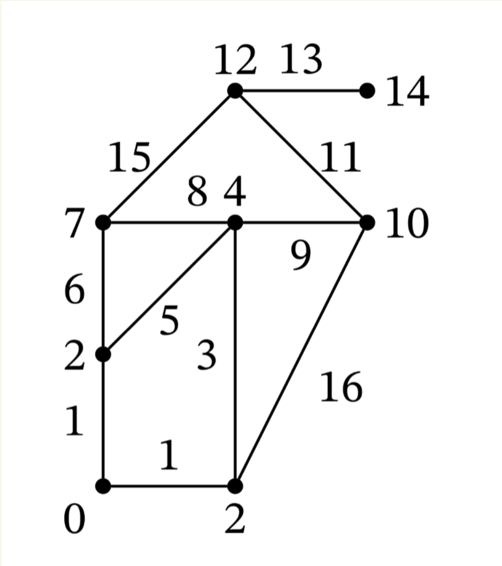


Regular points + discrete gradients

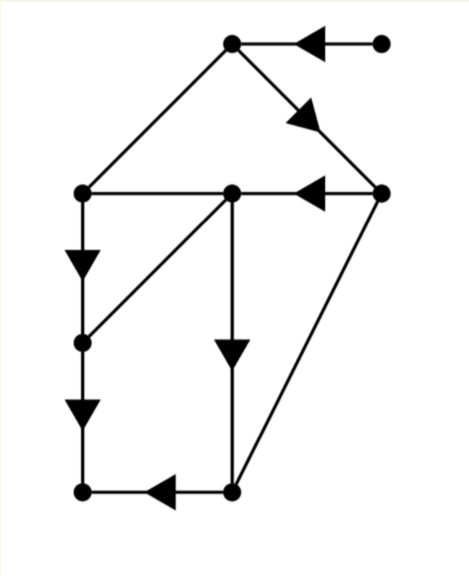
Any simplex that is not critical is regular, + will have one higher dim incident simplex with lower value **or** one lower dim simplex with higher.

Pair these!

(v, e)



$\Longrightarrow$



Exclusion lemma: Can't have both

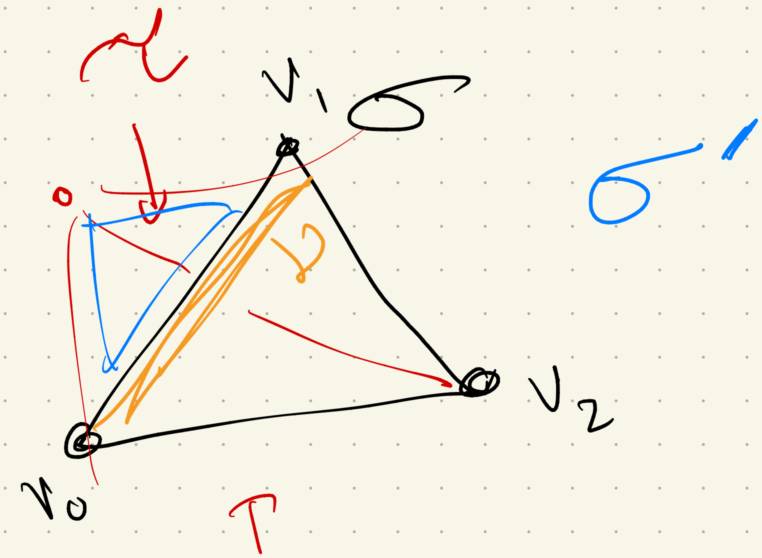
Proof: $\sigma = v_0 \ldots v_p$, & spps both:

$\tau = v_0 \ldots v_p v_{p+1}$, $\nu = v_0, \ldots, v_{p-1}$ & $f(\tau) \leq f(\sigma) \leq f(\nu)$

Let $\tilde{\sigma} = v_0, \ldots v_{p-1}, v_{p+1} \Rightarrow f(\nu) < f(\tilde{\sigma})$

& $f(\tilde{\sigma}) < f(\tau)$. But:

$f(\tau) \leq f(\sigma) \leq f(\nu) < f(\tilde{\sigma}) < f(\tau)$ ↯



Back to motivation

In Foreman's original work, goal was to identify a simpler complex with same homology:

Let $M_p \subseteq C_p(K)$ be critical p-simplices

Then $\exists$ maps $\tilde{\partial}_p$ s.t. $\tilde{\partial}_{d-1} \circ \tilde{\partial}_d = 0$

with $M_d \xrightarrow{\tilde{\partial}_d} M_{d-1} \xrightarrow{\tilde{\partial}_{d-1}} \cdots \xrightarrow{\tilde{\partial}_1} M_0$

s.t. $H_d(M, \tilde{\partial}) \cong H_d(K)$

(Uses CW complexes + homotopy equivalence)

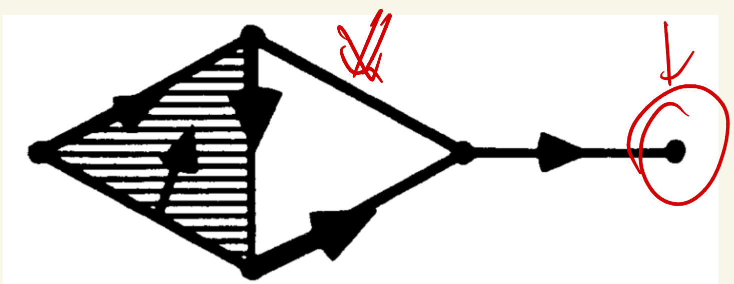
Theorem (Forman 1995)

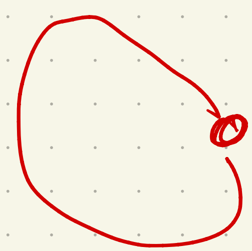
Suppose $\sigma^{(p)}$ is a critical simplex with $f(\sigma) \in [a,b]$, + there are no other critical simplices in $[a,b]$. Then,

all simplices σ with $f(\sigma) \leq b$

$M_{\leq b}$ is homotopy equivalent to $M_{\leq a} \cup e^{(p)}$, where $e^{(p)}$ is a p-cell glued to $M_{\leq a}$ along its boundary.

Can prove via collapses:



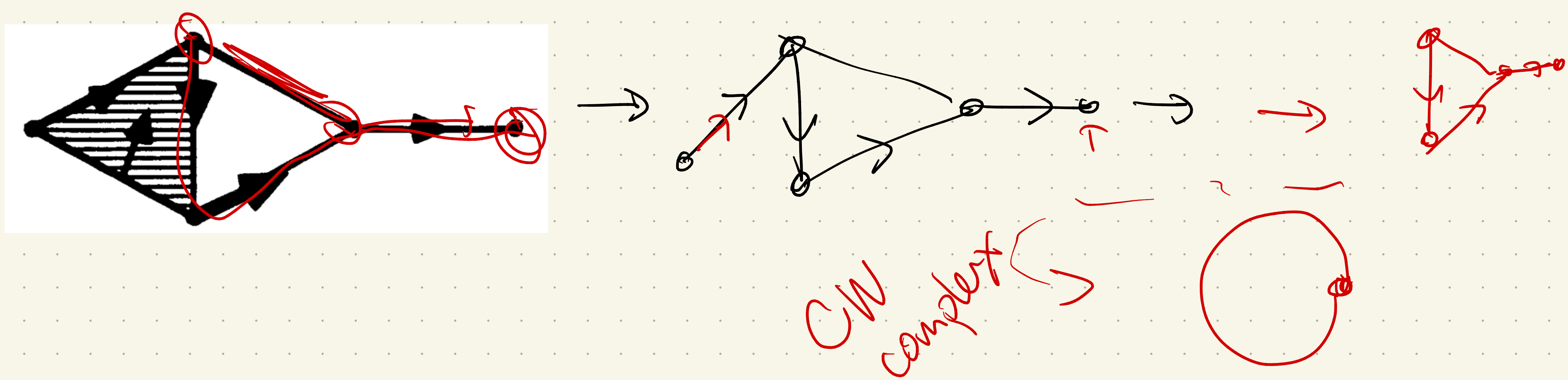


Why does this work?

Exclusion lemma!

Any pair is call an elemetary collapse $\longrightarrow$ won't change topology.

No conflicts, so end with simpler complex, ie



Discrete Morse inequalities

Let $f: K \to \mathbb{R}$ be a discrete Morse function with m_i critical ~~values~~ simplices in dim i, $i = 0..d$.

Then: $\beta_i(K) \le m_i \quad \forall i$ (a)

and $$\sum_{i=0}^{d} (-1)^i m_i = \underbrace{\chi(K)}$$

Euler characteristic: $\sum (-1)^i$ (# dim i cells)

$V - E + F$

Proof (a): induction on # of simplices

Base case: • $\quad \beta_0(K) = 1$ (& others $= 0$)

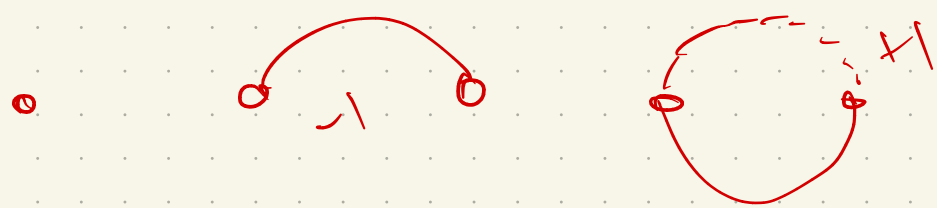
Inductive Step: Suppose true for any complex with $1 \le j \le \ell$ simplices.

Consider K with $\ell + 1$ simplices.

Let σ be maximum simplex of f, & K' be $K - \sigma$.

σ regular: σ pairs with some existing simplex $\rightarrow$ collapse & not change homology

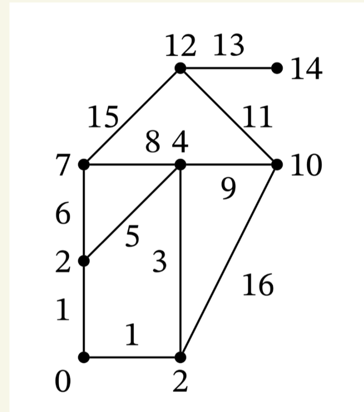
σ **critical:** dim p $\rightarrow$ changes $p-1$ & p homology



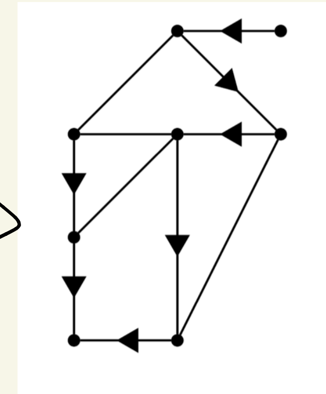
↳ added simplex, so ineq. still holds

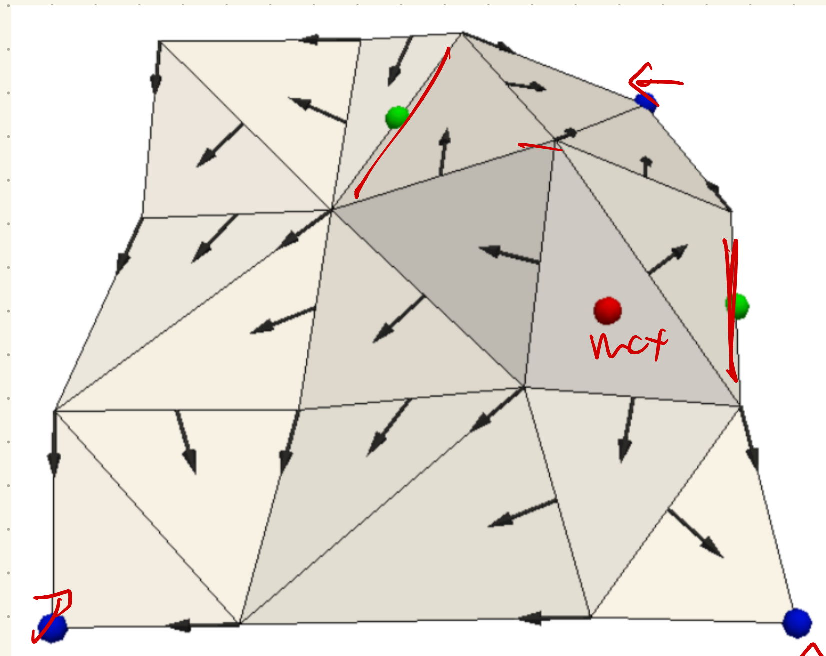
Result: Discrete Gradient Vector field:

Draw arrow from σ to higher dim nbr with lower value



⇒

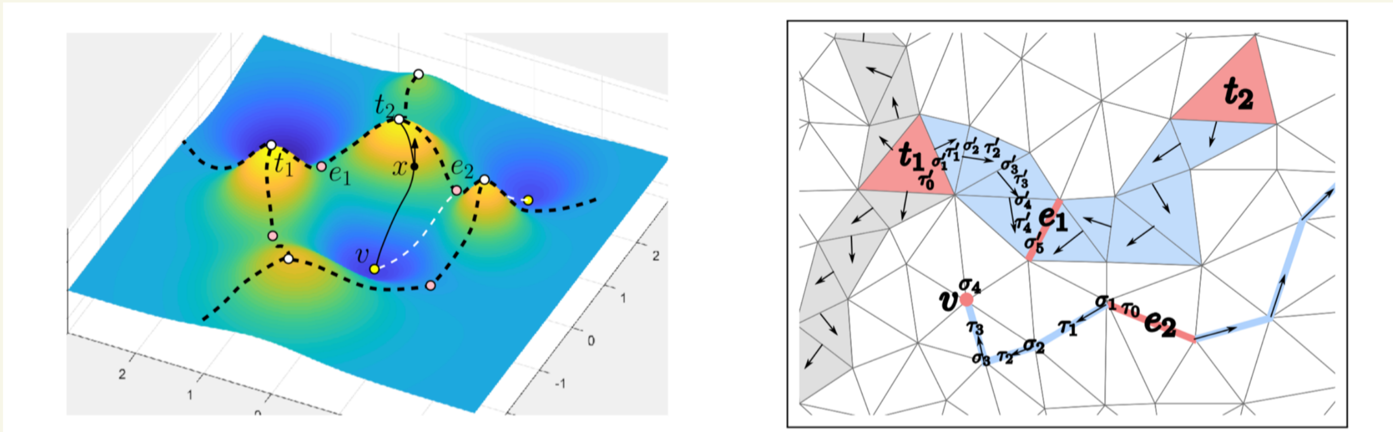




min

- Each simplex "flows" to at most one nbr
- Flow lines go down
- Flow vanishes at min critical simplices
  ↳ why? not paired

Back to V-paths: either face-edge
or edge-vertex

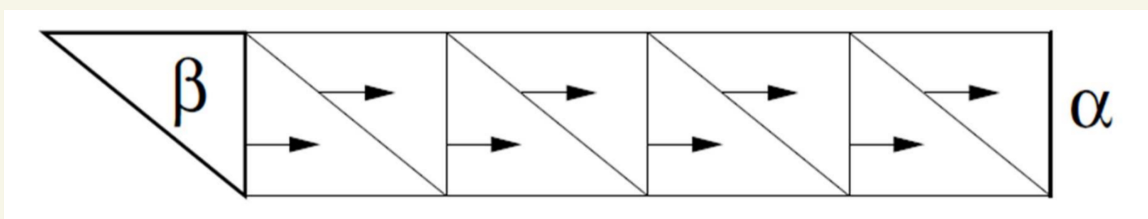


(a) (b)

continuous
flow lines

discretized
V-paths

ie:



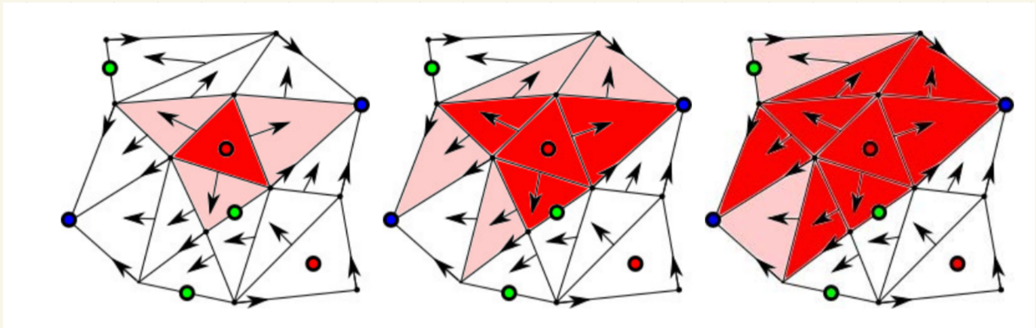
one difference: discrete flow goes down
(not up like continuous)

So, in discrete setting: For critical edge e:

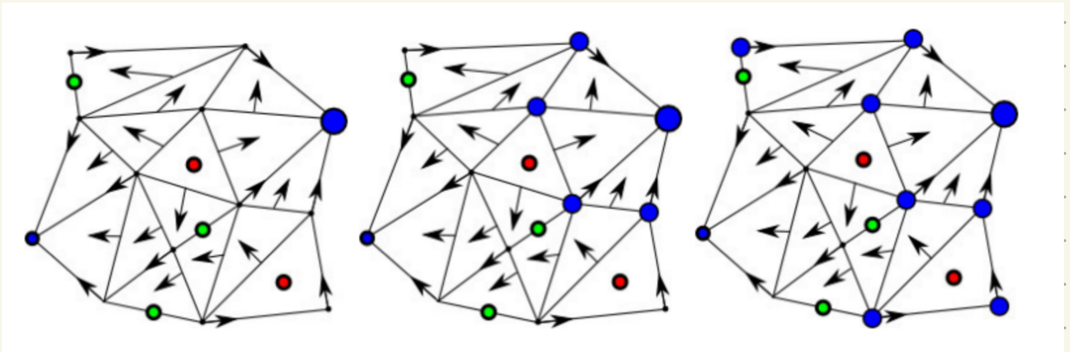
- stable manifold is union of edge-triangle gradient paths
- unstable manifold is union of vertex edge gradient paths

ie:

descending:

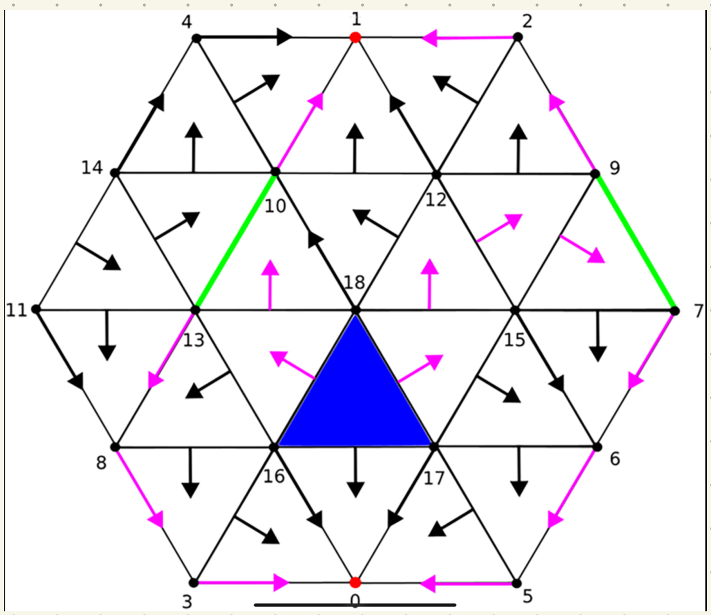


ascending:



Separatrices:

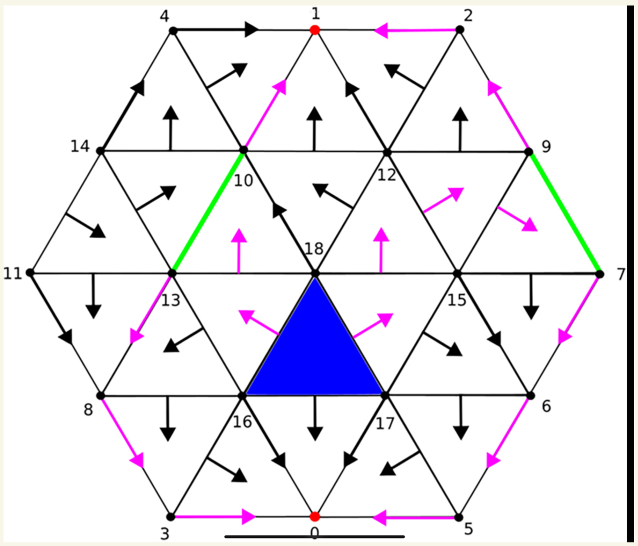
V-paths between critical simplices (marked pink)



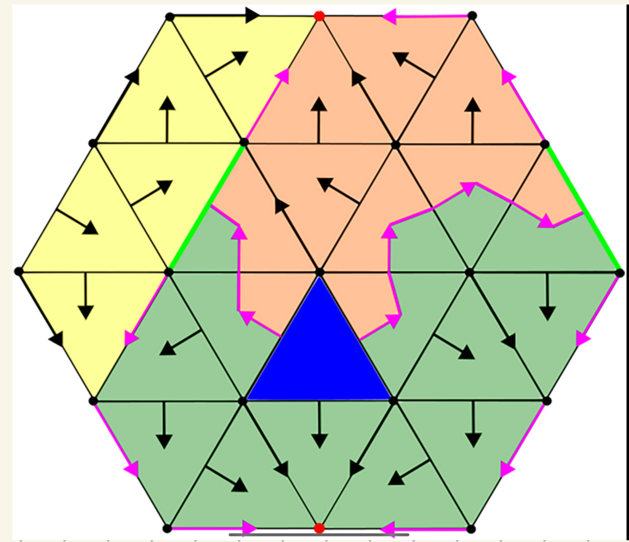
How to find?

- easy starting from critical edges
- From critical faces: try all options

Discrete Morse-Smale Complex

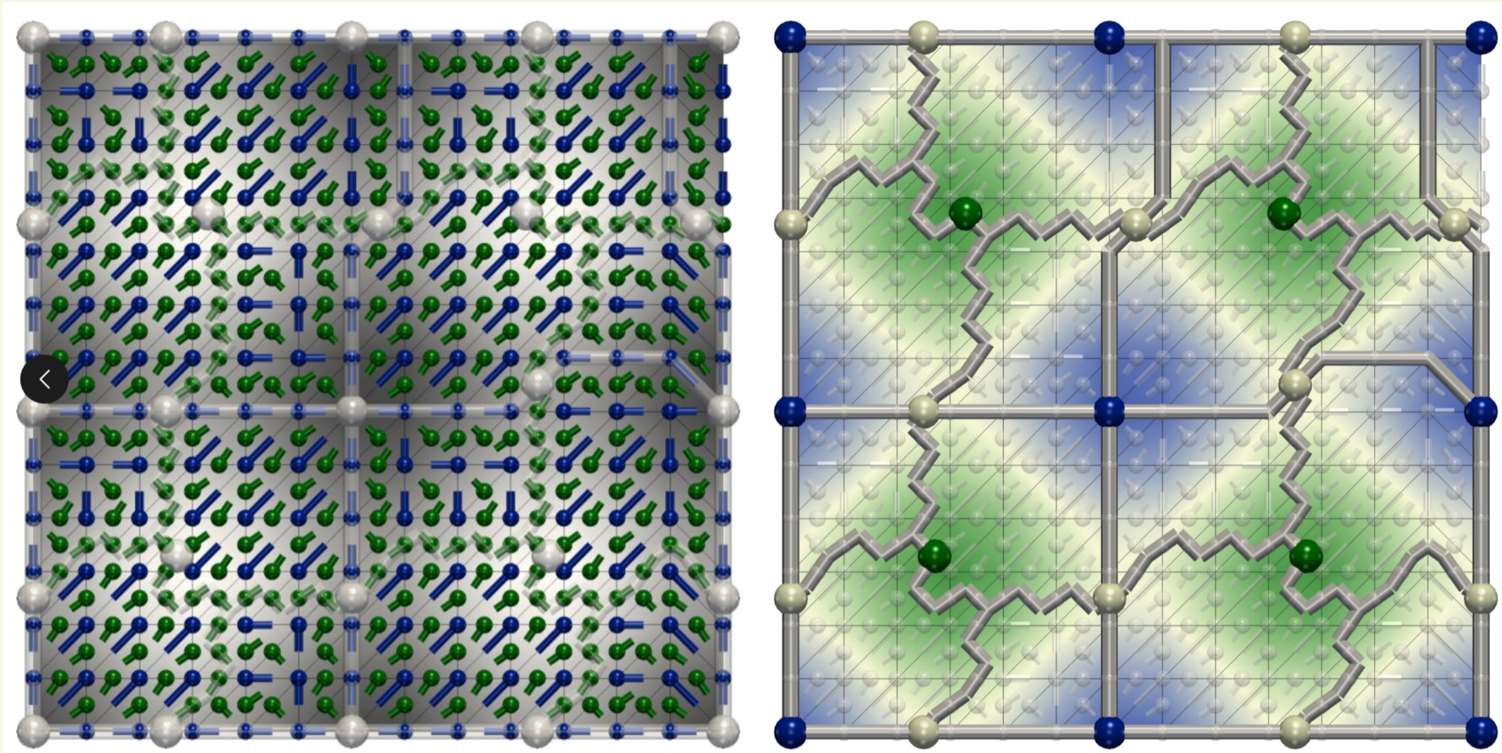


$\Rightarrow$

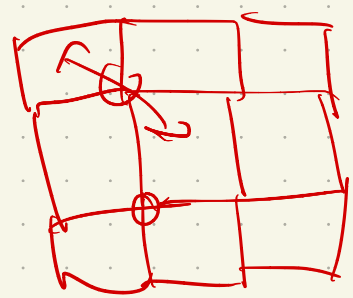


Algorithm: Collect ascending + descending manifolds of each critical simplex

Result: Partition of complex

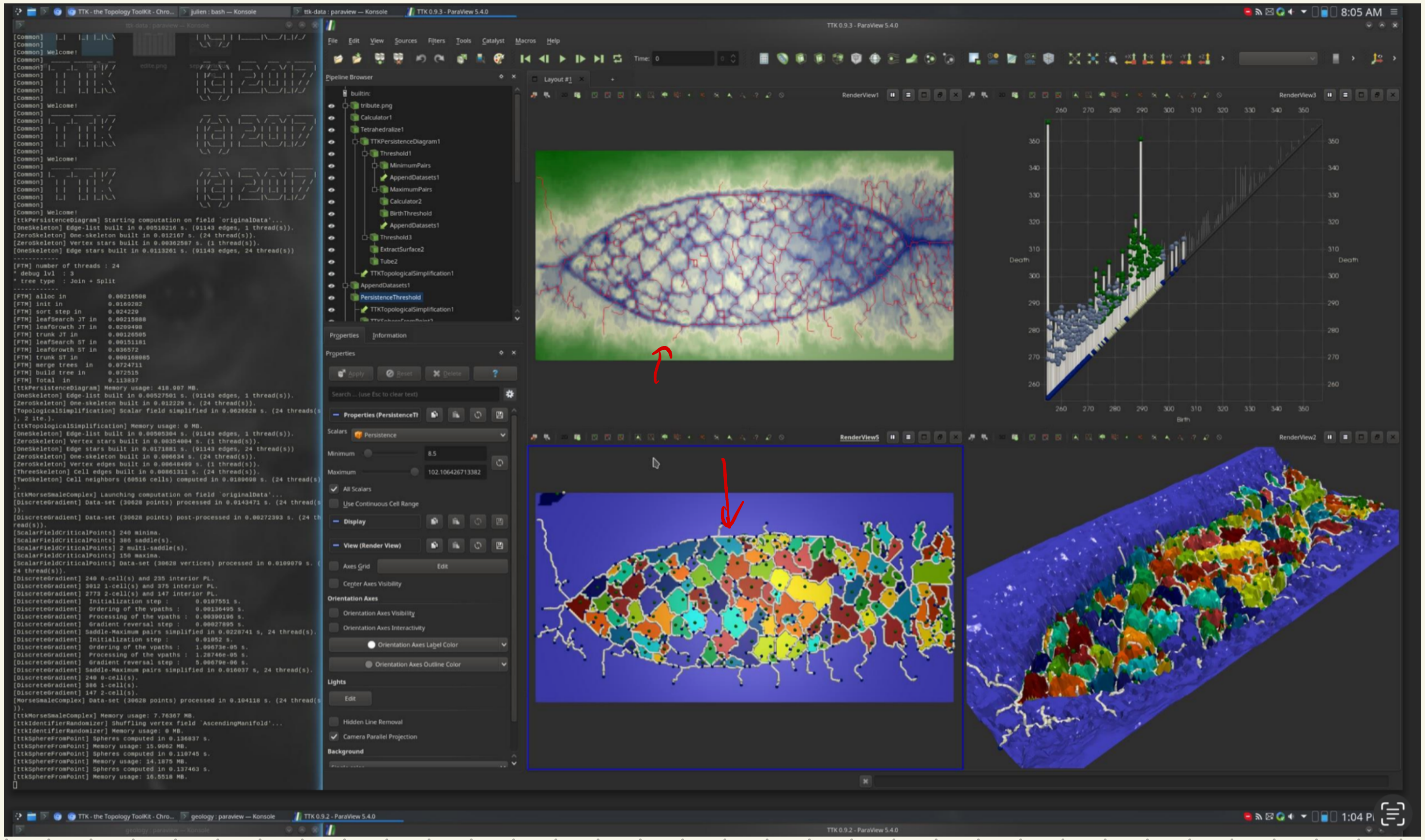


Caveat: only on 2-manifolds



In TTK → topology toolkit

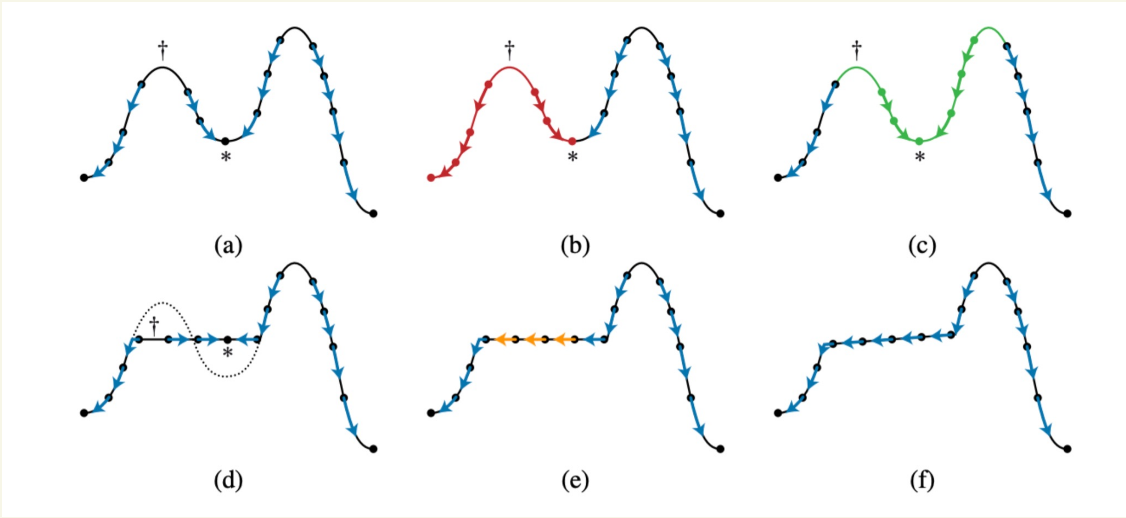
Methods to also apply this to greyscale data → partitioning & skeletonizing



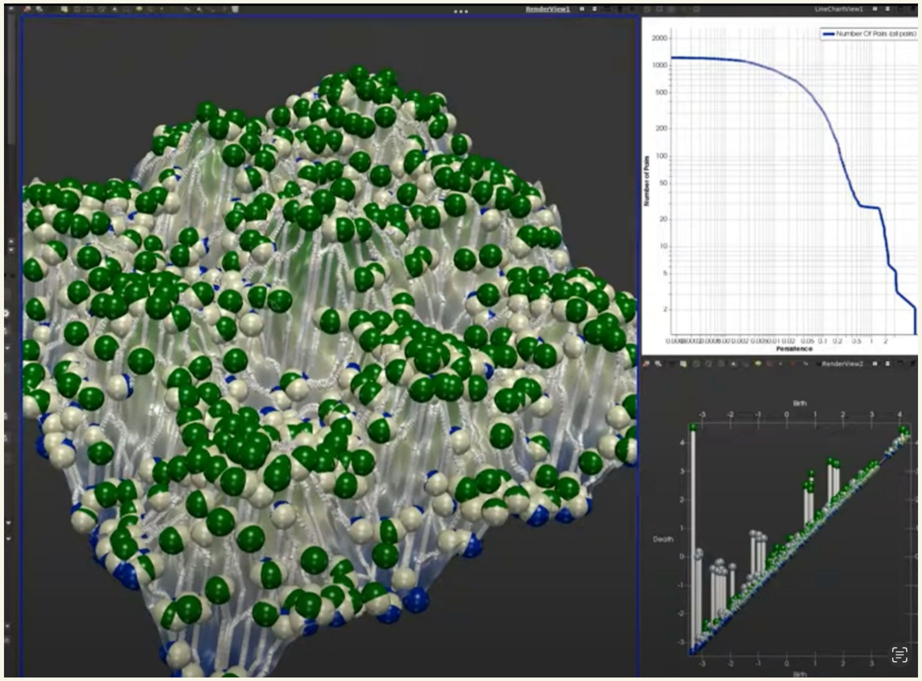
Persistence Simplifications

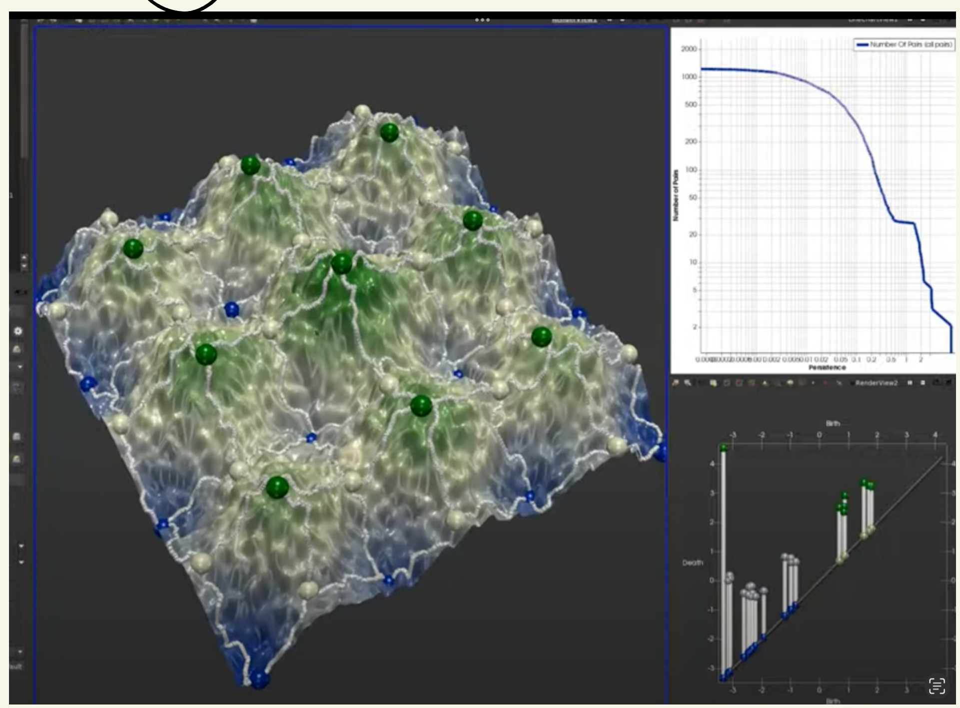
Bauer, Lange, Wardetzky 2010 proposed a method to simplify 1- + 2-manifolds using these ideas.

Focus on critical points!



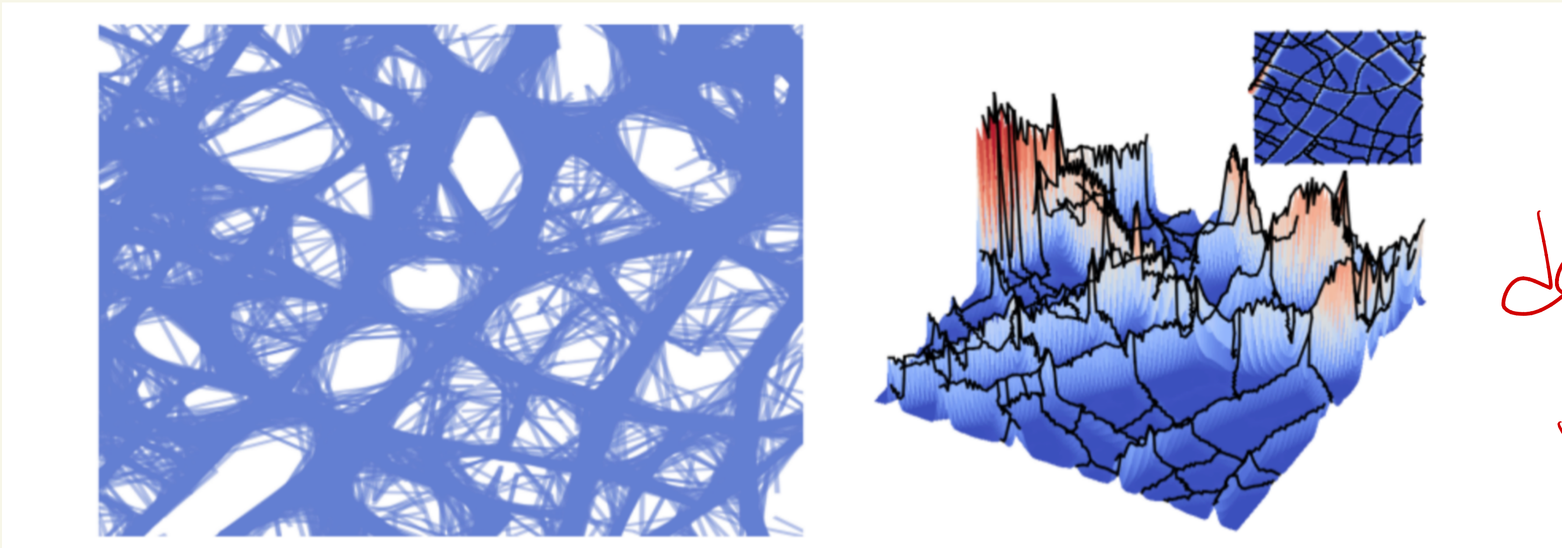
In action


Number Of Pairs (all pairs)
Number of Pairs
Persistence
Birth
Death


Number Of Pairs (all pairs)
Number of Pairs
Persistence
Birth
Death

Uses: Road network analysis

Wang, Wang & Li 2018

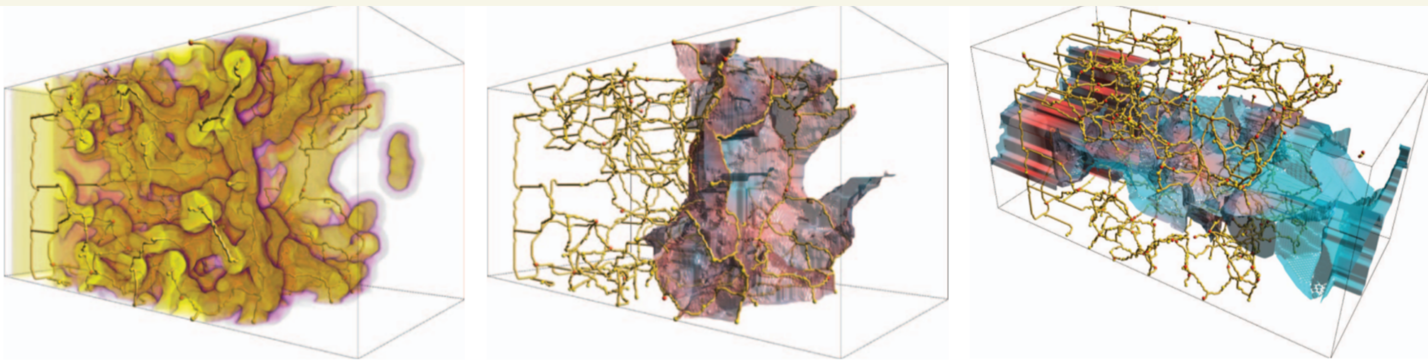


density as height

Goal: Reconstruct map from GPS data & Analyze traffic patterns

Another: Analyze porous solids

$\boxed{\text{Gyulassy et el 2012}}$



Porous Solid (left), $115 \times 115 \times 237$, float: A signed distance field from the material boundary represents a porous copper foam. This time-step ends a sequence where a micro-meteoroid impacts the foam from the right. We identify core structure, and additionally we extract a surface representing the impact crater (middle), and shear surface through the weakest points connecting the top half of the material to the bottom (right). *Impact Crater: (middle)* Simplify to 5.0% total persistence using the threshold values found by Gyulassy et al. [41]. Select 2-saddle-maximum arcs in the range $[-2.0, 30]$ and incident nodes for rendering. Select the lowest minimum (located outside the crater, and select neighboring 1-saddles (using two incidence selectors). Extract ascending 2-manifolds from these saddles, smooth, and apply a blue (low) to red (high) ramp transfer function with transparency. *Fracture Surface: (right)* Use the same hierarchy and arcs/nodes for rendering as in (Middle). Select all 2-saddle-maximum arcs that cross y-coordinate = 57.5. Select 2-saddles from their incident nodes. Extract descending 2-manifolds from these, smooth, and apply a blue (low) to red (high) transfer function with transparency.

Also see $\boxed{\text{Delgado-Friedricks et al 2015}}$

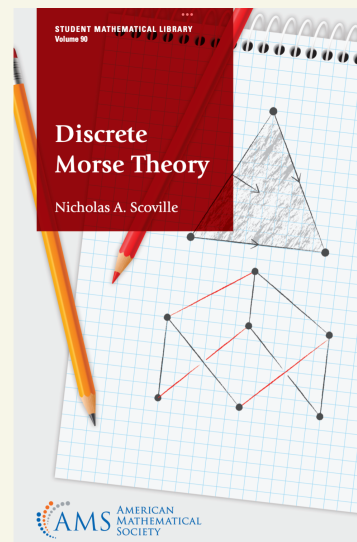
& $\boxed{\text{Robins et al 2011}}$

Also in textbook!

- Detailed algorithms for persistence cancellation

Recomendation if you want basics:

Discrete Morse theory by Scoville



Otherwise, go play with TTK data!