# Topological Data Analysis
## Fall 2025

Syllabus
Top. Review

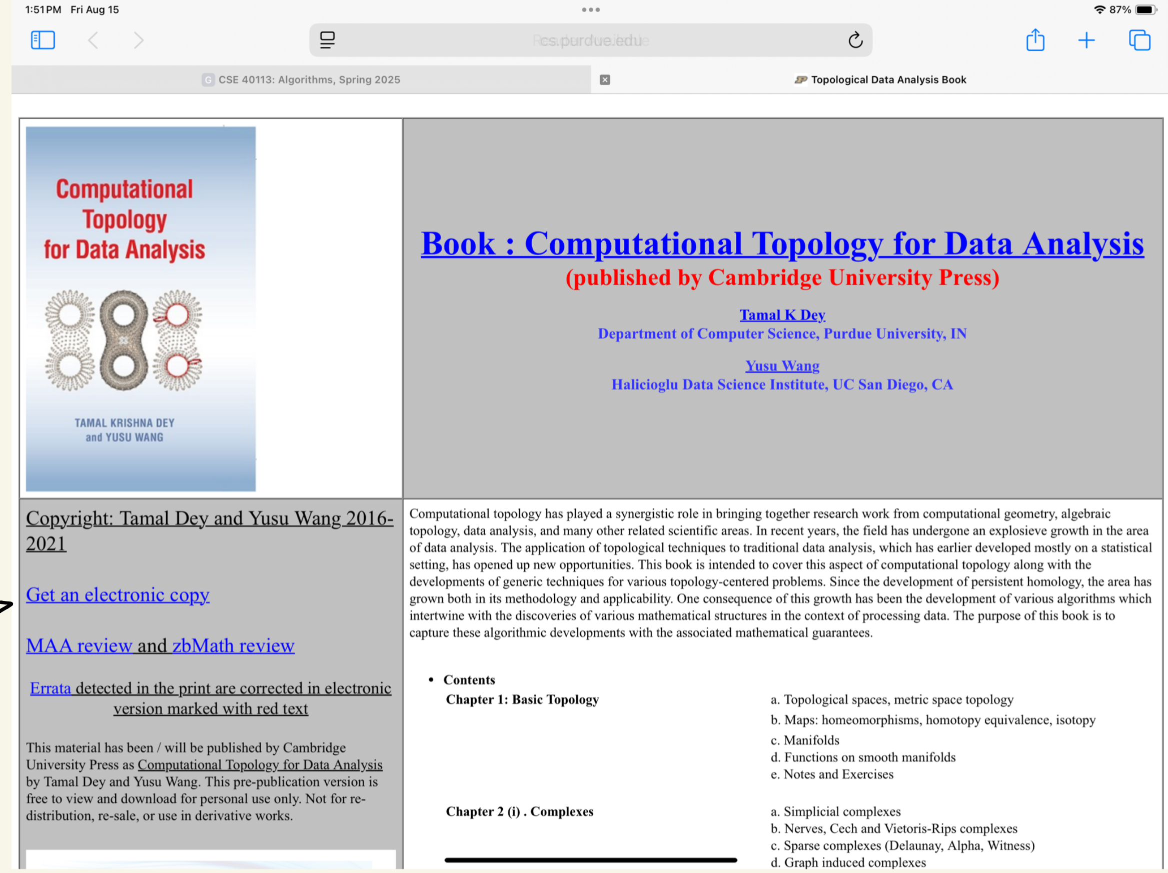
# Course intro

- Syllabus + HW → main webpage
- HW submission → Canvas
  (+ gradebook)
- Prereqs: some linear algebra
  some programming background
- Slack?
- HW: Mix of pen/paper + coding
  ↳ flexible, so see me if you have any issues!

# Textbook(s)

Main reference:

free pdf



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CSE 40113: Algorithms, Spring 2025

Topological Data Analysis Book

Computational Topology for Data Analysis

TAMAL KRISHNA DEY and YUSU WANG

## Book : Computational Topology for Data Analysis

**(published by Cambridge University Press)**

Tamal K Dey
Department of Computer Science, Purdue University, IN

Yusu Wang
Halicioglu Data Science Institute, UC San Diego, CA

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Get an electronic copy

MAA review and zbMath review

Errata detected in the print are corrected in electronic version marked with red text

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Computational topology has played a synergistic role in bringing together research work from computational geometry, algebraic topology, data analysis, and many other related scientific areas. In recent years, the field has undergone an explosieve growth in the area of data analysis. The application of topological techniques to traditional data analysis, which has earlier developed mostly on a statistical setting, has opened up new opportunities. This book is intended to cover this aspect of computational topology along with the developments of generic techniques for various topology-centered problems. Since the development of persistent homology, the area has grown both in its methodology and applicability. One consequence of this growth has been the development of various algorithms which intertwine with the discoveries of various mathematical structures in the context of processing data. The purpose of this book is to capture these algorithmic developments with the associated mathematical guarantees.

- **Contents**

| | |
|---|---|
| **Chapter 1: Basic Topology** | a. Topological spaces, metric space topology<br>b. Maps: homeomorphisms, homotopy equivalence, isotopy<br>c. Manifolds<br>d. Functions on smooth manifolds<br>e. Notes and Exercises |
| **Chapter 2 (i) . Complexes** | a. Simplicial complexes<br>b. Nerves, Cech and Vietoris-Rips complexes<br>c. Sparse complexes (Delaunay, Alpha, Witness)<br>d. Graph induced complexes |

Others:

Listed on Syllabus

Most are available in the library, or come visit my office!

Project:

Ideally, this will connect to your research.

But - we'll have several assignments to help explore topics, if you're unsure of what direction to take.

Outline:

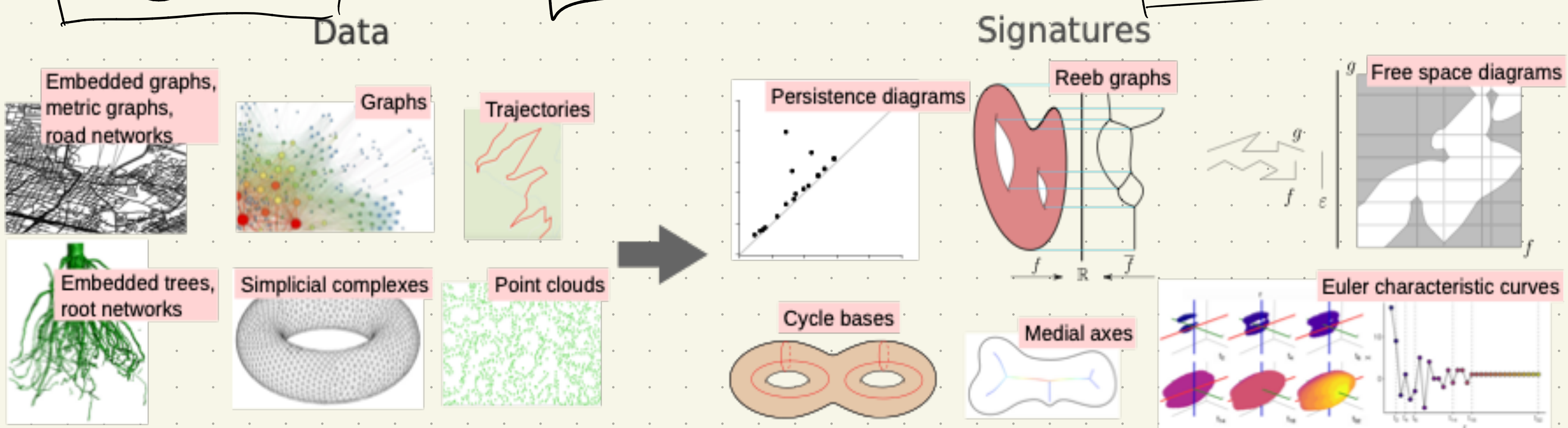
- talk summary → in Sept
- paper "chase"
- project proposal
- final presentation + submission

in October

# What is topological data analysis?

Traditional pipeline:

geometric data → topological signature → statistics, ML, visualization




Goals:

Concise
stable & robust
metrics

computable!

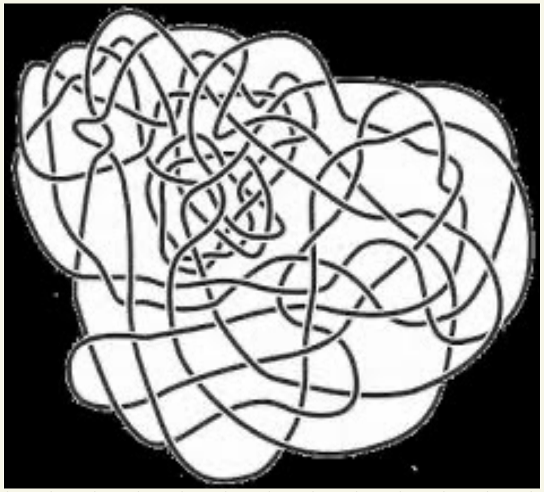
Some history!

Recognizing exact topology can be hard.

How so?

- Deciding if 2 4-manifolds are homeomorphic is undecidable [Markov 1960, van Meter 2005]

- Deciding if you can "unknot" a curve using a fixed number of moves is NP-Hard [de Mesmay, Sedgewick, & Tancer 2021]



## An approach

Since we can't solve the problem exactly, focus on invariants and ways to simplify the data.

This is not new!

Examples:

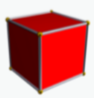
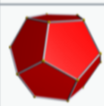
- Knot invariants
- Curve skeletons
- Manifold approximation
  - ↳ ie meshes

# A first example: Euler characteristic

Introduced first by Maurolico in 1537.

(This is known as

Then published by Euler in 1758:

| Name | Image | Vertices $V$ | Edges $E$ | Faces $F$ | Euler characteristic: $\chi = V - E + F$ |
|---|---|---|---|---|---|
| Tetrahedron |  | 4 | 6 | 4 | **2** |
| Hexahedron or cube |  | 8 | 12 | 6 | **2** |
| Octahedron |  | 6 | 12 | 8 | **2** |
| Dodecahedron |  | 20 | 30 | 12 | **2** |
| Icosahedron |  | 12 | 30 | 20 | **2** |

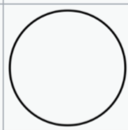
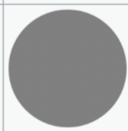
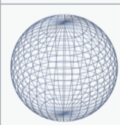
For any embedded planar graph $G=(V,E)$ with $F$ faces,

$$V-E+F=2$$

Note: planar $\Rightarrow$ embedded on sphere

## More generally:

Euler characteristic: $V - E + F = \chi$

| Name | Image | χ |
|---|---|---|
| Interval |  | 1 |
| Circle |  | 0 |
| Disk |  | 1 |
| Sphere |  | 2 |
| Torus *(Product of two circles)* |  | 0 |
| Double torus |  | −2 |
| Triple torus |  | −4 |
| Real projective plane |  | 1 |
| Möbius strip |  | 0 |

planar

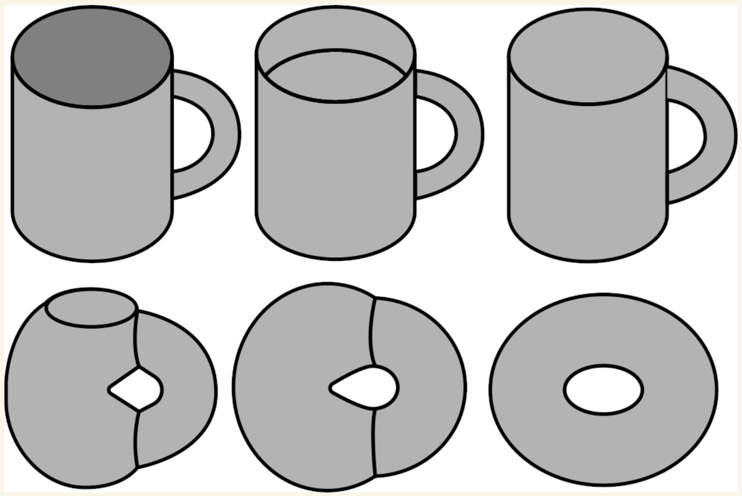
Ideal for computers:

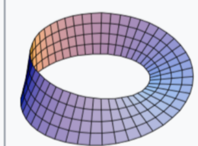
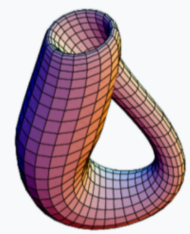
- requires a discrete representation
- Given a data structure encoding V, E, + F, easy to calculate

$V - E + F$

Topological signatures: invariants

Different Euler characteristic $\Rightarrow$ different space



| | | |
|---|---|---|
| Möbius strip |  | 0 |
| Klein bottle |  | 0 |

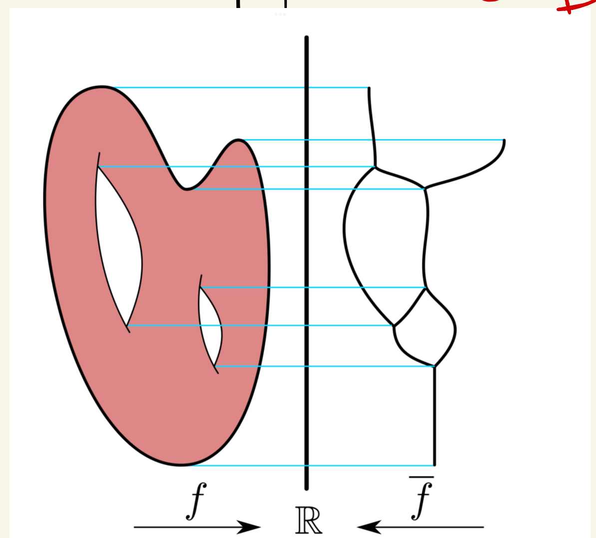
But different spaces might have the same Euler characteristic (as well as very different geometry)

# Back to signatures

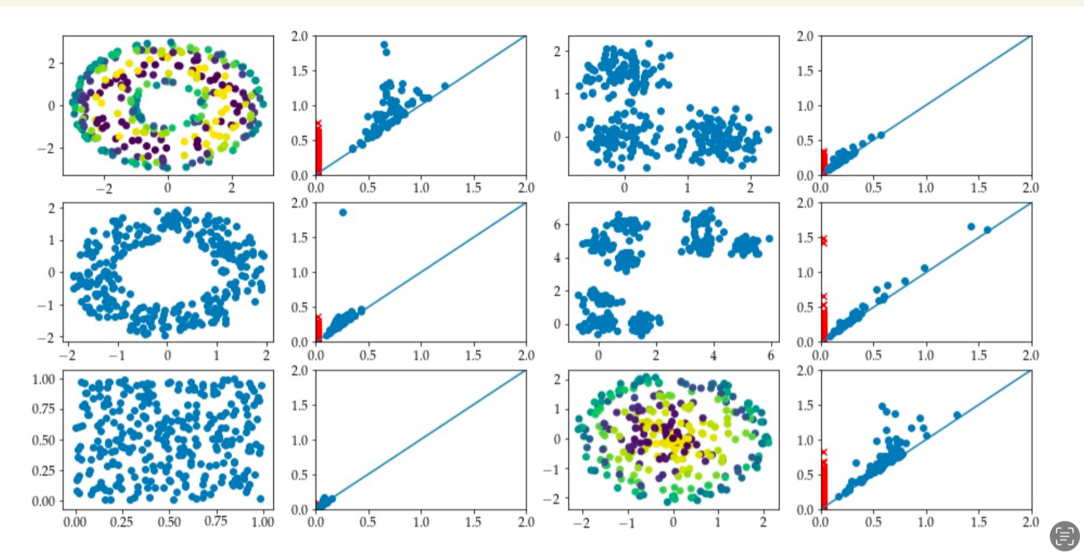
We'll cover a range of possible choices, on a sliding scale of complexity + discriminativity.

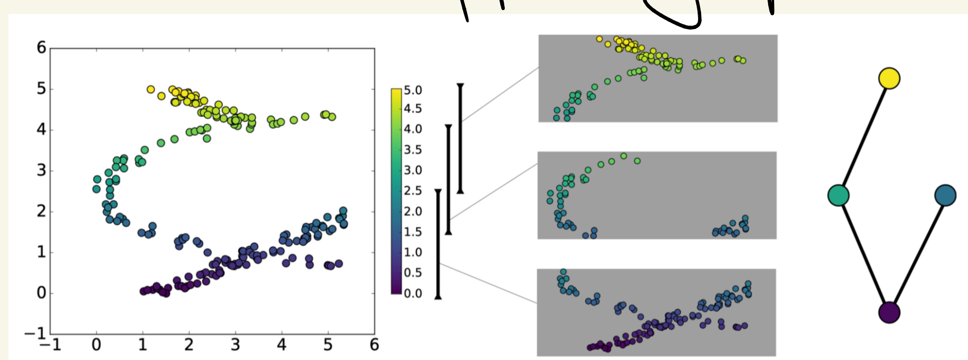
graph-based

Examples:




Reeb + Mapper graphs





Persistent homology

Algebraic topology

# Active research directions

This is a fairly young & vibrent area.

Emerging directions:

- Machine Leering with TDA
- Time series & dynamic systems
- Parellelization
- Vizualization
- Algebraic methods & multi-dimensional persistence

- Many applications: atmospheric data, image processing, biomedical, neuroscience, vision, etc.

# Our goals

Understand the computation + interpretation of several commonly used tools in TDA:

- Euler curves
- Persistent homology
- Mapper + Reeb graphs
- Morse-Smale Complexes

In the end, understand what types of signatures are likely to be both useful + practical on data sets (as well as which open source tools exist).

But first – topology!

Chapter 1 covers an intro to topology.

Depending on math, might seem obvious, or might seem very hard!

Either is ok.

Worth reading textbook to be sure you get main definitions in context, + come see me if you have questions.

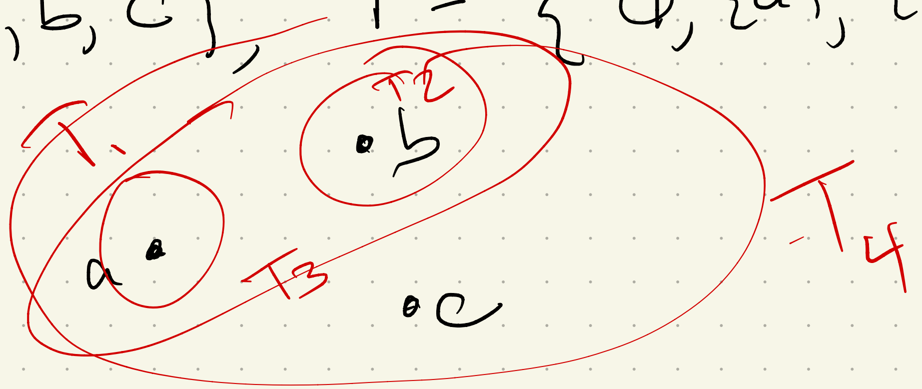
# Topology

**Topological space**: a set $\mathbb{T}$ with elements (called points) + a set of subsets $T$, such that

- $\phi, \mathbb{T} \in T$
- $\forall\, U \subseteq T$, union of sets in $U$ is in $T$
- $\forall$ finite $U \subseteq T$, intersection of sets in $U$ is also in $T$

Ex: $\mathbb{T} = \{a, b, c\}$, $T = \{\phi, \overset{T_1}{\{a\}}, \overset{T_2}{\{b\}}, \overset{T_3}{\{a,b\}}, \overset{T_4}{\{a,b,c\}}\}$

Check:



## Metric space:

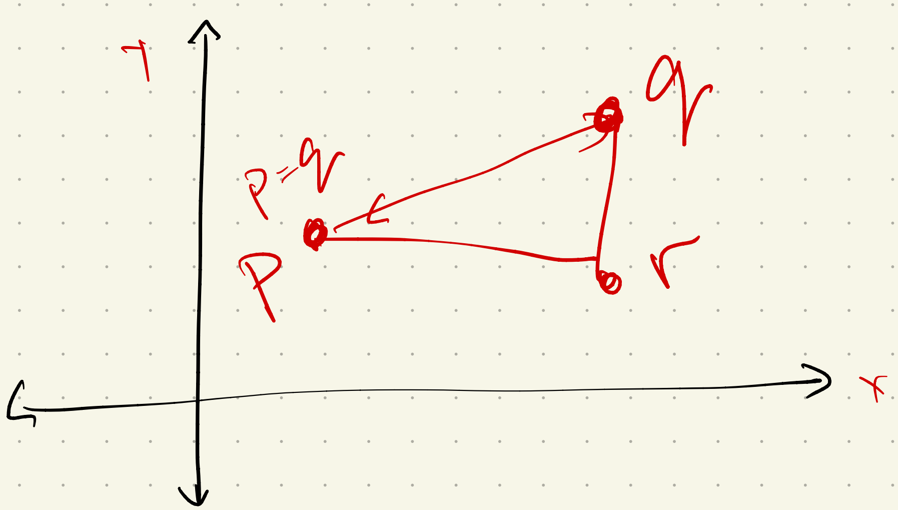
a pair $(\mathbb{T}, d)$, where $\mathbb{T}$ is a set and $d : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfies

other: $d(p,q) \geq 0$

- $d(p,q) = 0 \iff p = q$
- $d(p,q) = d(q,p) \quad \forall p,q \in \mathbb{T}$

triangle inequality:
- $d(p,q) \leq d(p,r) + d(r,q) \quad \forall p,q,r \in \mathbb{T}$

**Example:** $\mathbb{T} = \mathbb{R}^2$, $d((u_1,u_2),(v_1,v_2)) = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2}$




# Metric topology

Given a metric space $(\mathbb{T}, d)$, an open 


metric ball is

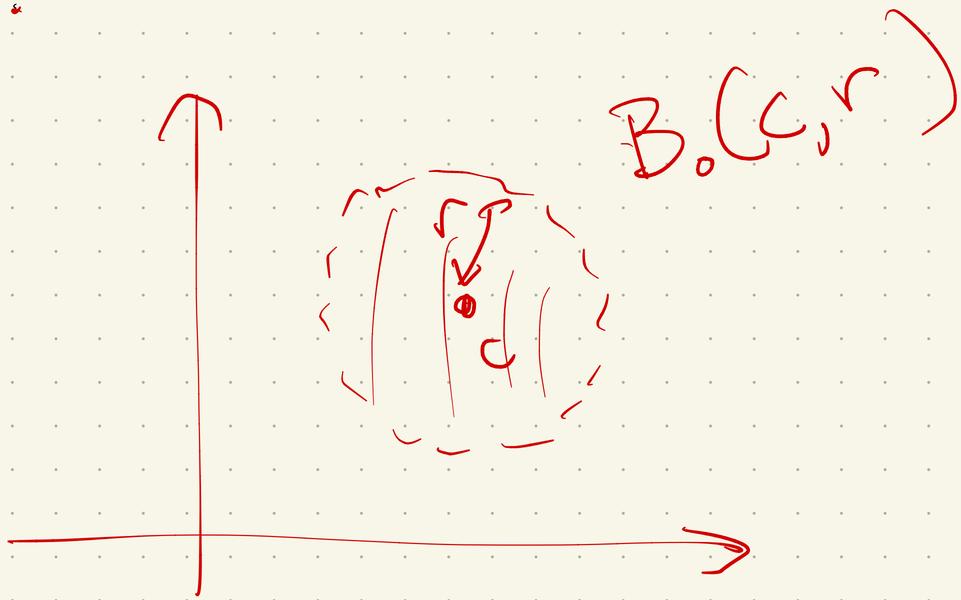
center

$$B_o(c,r) = \{\, p \in \mathbb{T} \mid d(p,c) < r \,\}$$

open

The metric topology is the set of all metric balls.

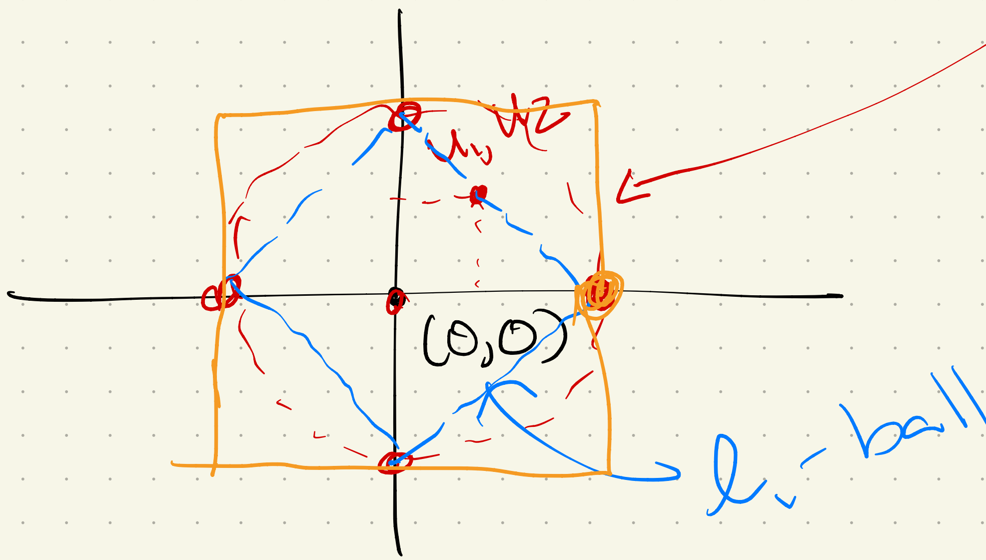
Ex: $\mathbb{R}^2$ again:



Many different metric topologies!

Fix $\mathbb{R}^2$, + let's try $B_0(0,1)$ for:

- $\|u-v\|_1 = |u_1 - v_1| + |u_2 - v_2| < 1$ ? $\ell_1$ metric
- $\|u-v\|_2 = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2}$ $\ell_2$-metric
- $\|u-v\|_\infty = \max\{|u_1-v_1|, |u_2-v_2|\}$ $\ell_\infty$-metric




$\ell_p$-metrics

# Open & closed sets

Fixing a topology $T$, $U$ is open if $U \in T$.

We say $U$ is closed if $\mathbb{I} \setminus U$ is open.

set theory complement

Back to first example:

$\mathbb{I} = \{a, b, c\}$, $T = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$

open

Closed sets:

$\mathbb{I} - \emptyset = \mathbb{I}$

$\mathbb{I} - \{a\} = \{b, c\}$

$\mathbb{I} - \{b\} = \{a, c\}$

$\mathbb{I} - \{a, b\} = \{c\}$

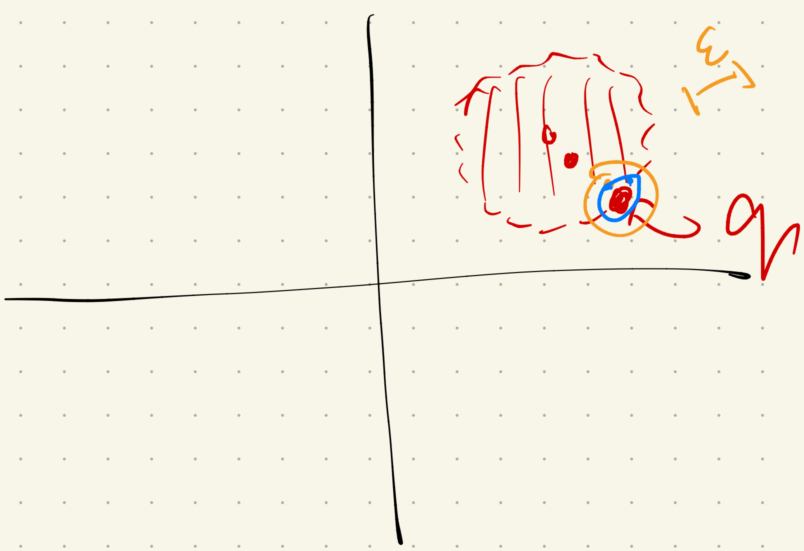
$\mathbb{I} - \{a, b, c\} = \emptyset$

In a metric space, can get some alternate definitions:

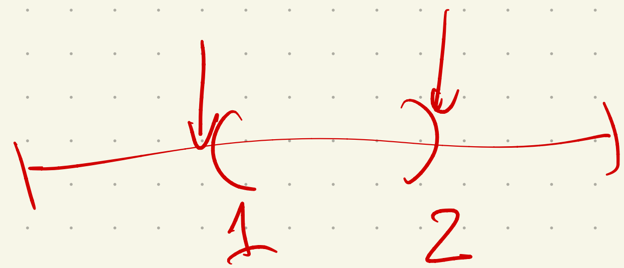
Consider $\boxed{Q \subseteq \mathbb{T}}$. A point $p \in \mathbb{T}$ is a **limit point** of $Q$ if $\forall \varepsilon > 0$, $Q$ contains some point $q \neq p$ with $d(p,q) < \varepsilon$

Example: $\mathbb{R}^2$ & $B_0(p, 1)$:

distance $< 1$




$[1,2]$ vs $(1,2)$




$(1,2) \subset \mathbb{R}$

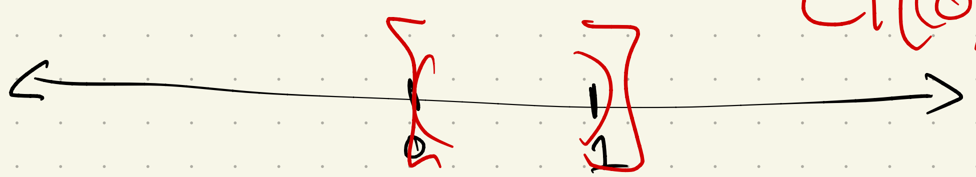
The **closure** of a point set $Q \subseteq \mathbb{T}$ is the set containing $Q$ and all of its limit points.

↳ written $Cl(Q)$, or $\overline{Q}$

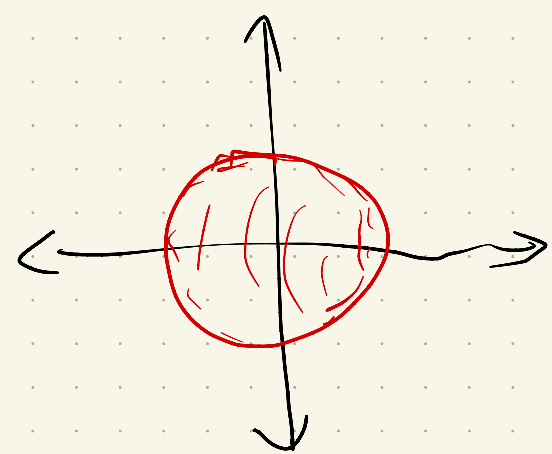
We say $Q$ is **closed** if $Q = Cl(Q)$.

Example: $(0,1) \subseteq \mathbb{R}$ is an open ball

$Cl((0,1)) = [0,1]$



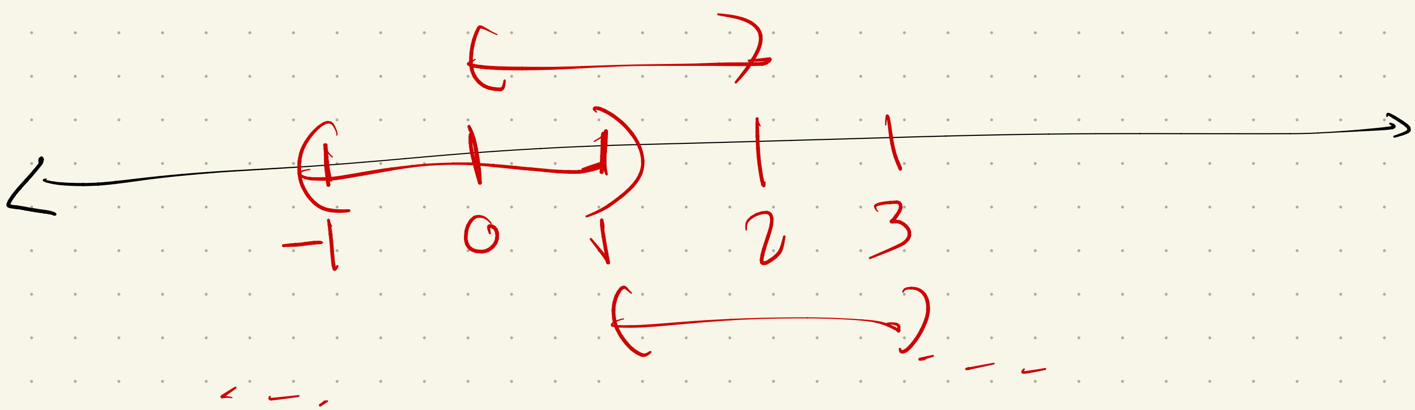

Example: $\mathbb{R}^2$ + $B_o(0,1)$



A open (resp, closed) cover of a topological space $(\mathbb{T}, \mathcal{T})$ is a collection $C$ of open (resp. closed) sets s.t.

$$\mathbb{T} = \bigcup_{c \in C} c$$

Example: $\mathbb{R}$, $C = \{ (n-1, n+1) \mid n \in \mathbb{Z} \}$



A topological space is disconnected if $\exists$ 2 disjoint nonempty open sets $U, V \in \mathcal{T}$ s.t. $\mathbb{T} = U \cup V$.

(The space is connected if it is not disconnected.)

Ex: $A = (1,2) \cup (3,4) \subset \mathbb{R}$

Note: subspace topology: Given $\mathbb{U} \subseteq \mathbb{T}$, $\mathbb{U}$ can inherit topology from $\mathcal{T}$ via

$$\{X \cap U \mid X \in \mathcal{T}\}$$

today 1.1 & 1.2

Next time:

Maps, homeomorphisms, & homotopies!

(See remainder of Chapter 1)

Overall goal: understand enough about maps to get to "nice" functions, & start Morse theory

Homework 0: Send me an email!

(HW 1 also posted, but you have 2 weeks.)