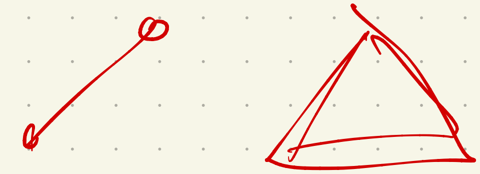
TDA - fall 2025

Induced &
relative
homology

Last time: Homology!



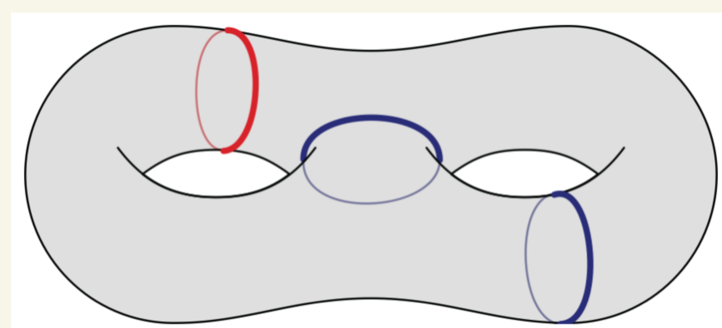
$\mathbb{Z}_2$

$$\cdots \to C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \to \cdots$$

$$H_p(K) = \overset{\text{cycles}}{Z_p} \Big/ \underset{\text{boundaries}}{B_p} = \ker \partial_p \big/ \operatorname{im} \partial_{p+1}$$

Why?

- Computable! & homologous cycles are somehow "the same"
- But → <u>not</u> homotopy or isotopy.



Computing homology groups

To compute Betti number:

$$\beta_p = \dim(H_p(K))$$

Well, for any linear transformation $f: U \to V$,

$$\dim(U) = \dim(\ker f) + \dim(\operatorname{im} f)$$

set $f = \partial_p$: $\quad \ker \partial_p \quad \operatorname{im} \partial_p$

$$\dim(C_p) = \dim(Z_p) + \dim(B_{p-1})$$

Also, for a quotient space V/W,

$$\dim(V/W) = \dim(V) - \dim(W)$$

$$\Longrightarrow \beta_p = \dim(Z_p) - \dim(B_p)$$

So: computing!

Back to boundary matrices:

$$\partial_p \cdot \underbrace{\alpha}_{p} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n_p} \\ b_{2,1} & & & b_{2,n_p} \\ \vdots & & & \\ b_{n_{(p-1)},1} & \cdots & & b_{n_{p-1},n_p} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_{n_p} \end{bmatrix}$$

$= p\text{-}1$ chain

where

$$\alpha = \sum_{\substack{\text{all } p\text{-simplices} \\ \sigma_i}} a_i \sigma_i$$

Rows are a basis for C_{p-1}

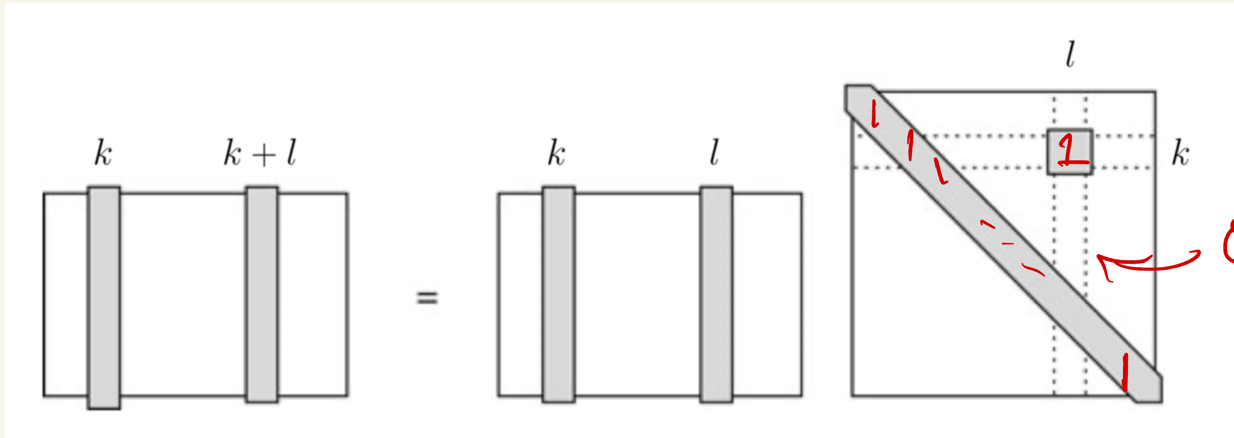
Columns are a basis for C_p

How to find rank?

Operations on matrices

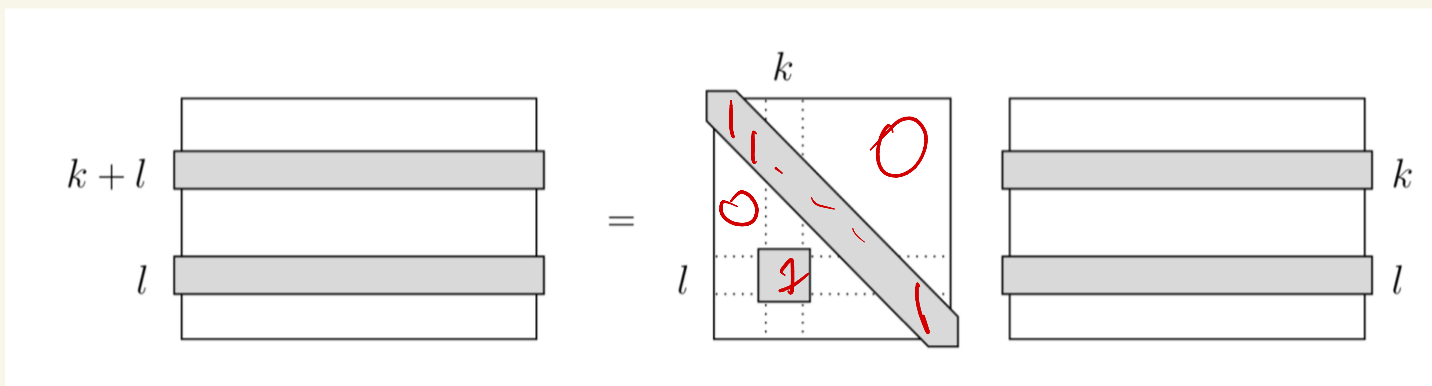
Simplify to Smith-Normal form. How?

Add columns



0 all other spots

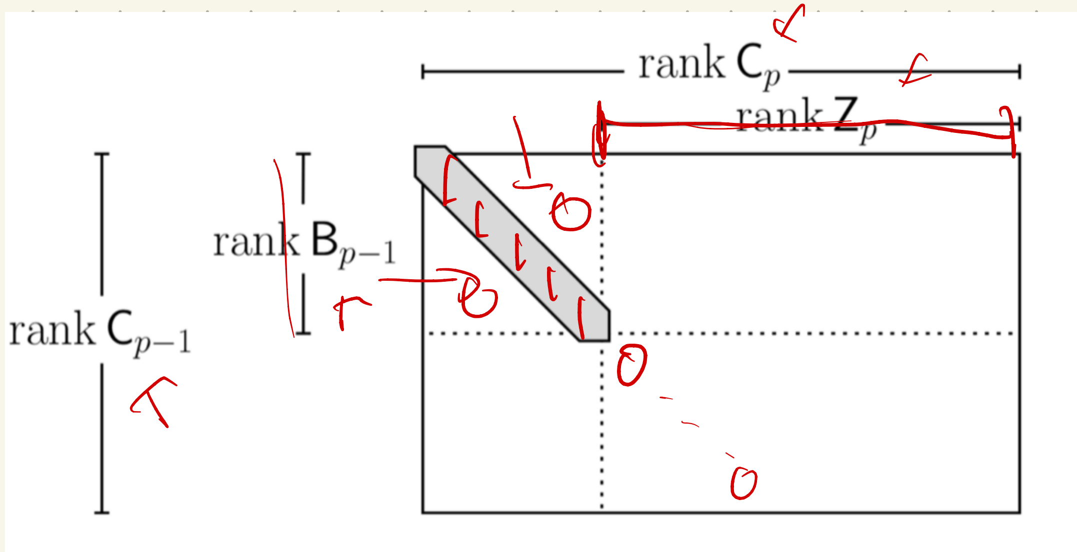
Add rows



(& exchange rows/columns with 0 on diagonal)

Goal: Move 1's to diagonal

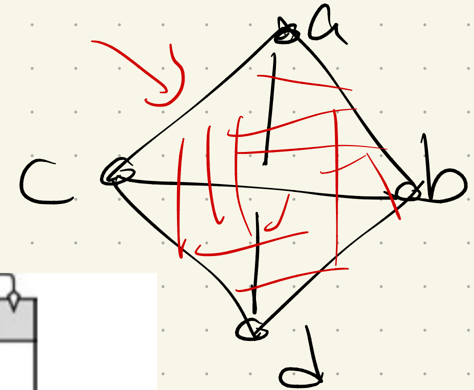
Smith-Normal form: $N_p = U_{p-1} \partial_p V_p$

$N_p =$ 

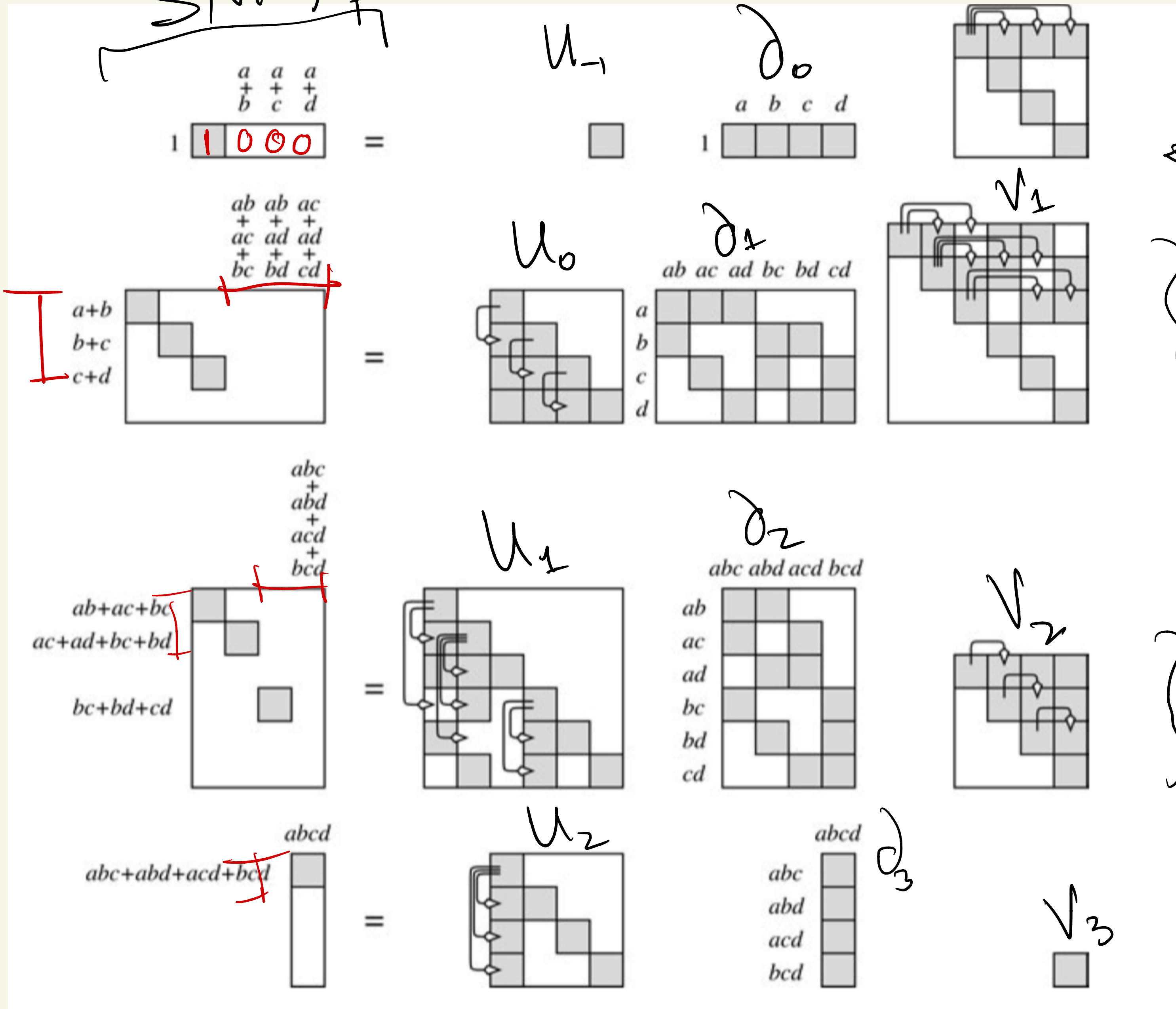
then $\beta_p = \text{rank}(Z_p) - \underbrace{\text{rank}(B_p)}_{?}$

$N_{p+1} =$ I ← rank B_p

An example: solid tetrahedron



SNF N_p



rank $Z_0 = 3$

rank $B_0 = 3$

rank $Z_1 = 3$

rank $B_1 = 2$

rank $Z_2 = 1$

rank $B_2 = 1$

$\sim$ rank $Z_0 = 3$

rank $B_0 = 3$

rank $Z_1 = 3$

rank $B_1 = 2$

rank $Z_2 = 1$

rank $B_2 = 1$

$\beta_0 = \text{rank } Z_0 - \text{rank } B_0$

$= 0$

$\beta_1 = \text{rank } Z_1 - \text{rank } B_1$

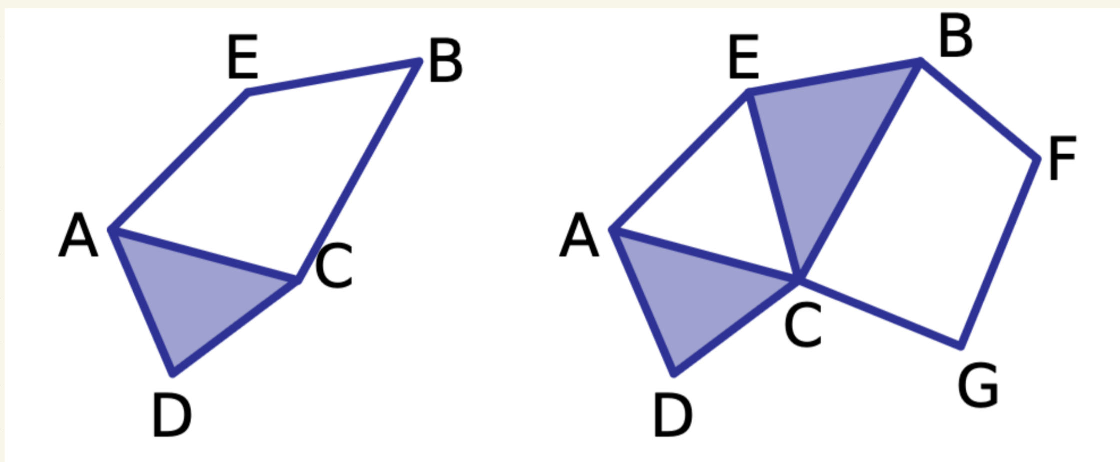
$= 3 - 2 = 1$

$\vdots$

Recall: A simplicial map between abstract simplicial complexes $f: K \to L$ is induced by a map on vertices $V(K) \to V(L)$

Inclusion maps: $i: K \to L$, $K \subseteq L$

$i(\sigma) = \sigma$



K $\qquad\qquad$ L

Passing to chain complexes

Any inclusion map $f: K \to L$ naturally extends to a map on chain complexes:

$$f_\#: C_p(K) \to C_p(L)$$

Results in a diagram:

$\alpha = \sum a_i \sigma_i$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1}(K) \arrow[r, "\partial^K_{p+1}"] \arrow[d, "f_\#"] & C_p(K) \arrow[r, "\partial^K_p"] \arrow[d, "f_\#"] & C_{p-1}(K) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{p+1}(L) \arrow[r, "\partial^L_{p+1}"'] & C_p(L) \arrow[r, "\partial^L_p"'] & C_{p-1}(L) \arrow[r] & \cdots
\end{tikzcd}$$

$\sum a_i f(\sigma_i)$

Claim: $f_\# \partial^K = \partial^L \circ f_\#$.

Why? remember how f worked on vertices.

$$\begin{tikzcd}
\alpha \in C_p(K) \arrow[r, "\partial_p^K"] \arrow[d, "f_\#"'] & C_{p-1}(K) \arrow[d, "f_\#"] \\
f_\#(\alpha)\, C_p(L) \arrow[r, "\partial_p^L"] & C_{p-1}(L)
\end{tikzcd}$$

$\partial_p^K(\alpha)$

$f_\#(\partial_K^k(\alpha))$

$//$

$\partial_p^L(f_\#(\alpha))$

Consider a $\alpha \in C_p(K)$.

Commutative diagram

$$\alpha = \sum a_i \cdot \sigma_i \longrightarrow \sum \partial(\sigma_i)$$

$\downarrow f_\#$ $\quad \underbrace{\sum a_i f(\sigma_i)}$ $\longrightarrow$ $f(\sigma_i)$'s boundaries

$\downarrow f_\#$ $\quad f_\#(\partial \sigma_i)$

Claim: $f_\#$(cycle in K) ~~=~~ is a cycle in L

$f_\#$(boundary in K) is a boundary in L

Why?

Because it commutes!

Consider a cycle: $\partial_p^K(\alpha) = 0$

$$\begin{tikzcd}
C_p(K) \arrow[r, "\partial_p^K"] \arrow[d, "f_\#"'] & C_{p-1}(K) \arrow[d, "f_\#"] \\
C_p(L) \arrow[r, "\partial_p^L"] & C_{p-1}(L)
\end{tikzcd}$$

α

0

0

$f(\alpha) - \Sigma a_i f_\#(\sigma)$

also cancels

This induces a map on homology:

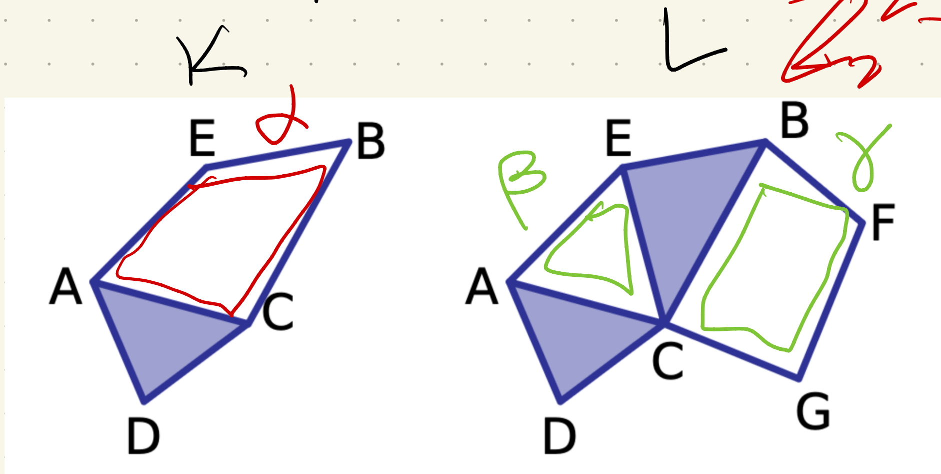
$$f_*: H_p(K) \longrightarrow H_p(L)$$

$$[\alpha] \longmapsto [f_\#(\alpha)]$$

$$\mathbb{Z}_2 = H_1(K) = \{ \emptyset, [\alpha] \}$$

$$\mathbb{Z}_2^2 = H_1(L) = \{ \emptyset, [\beta], [\gamma] \}$$

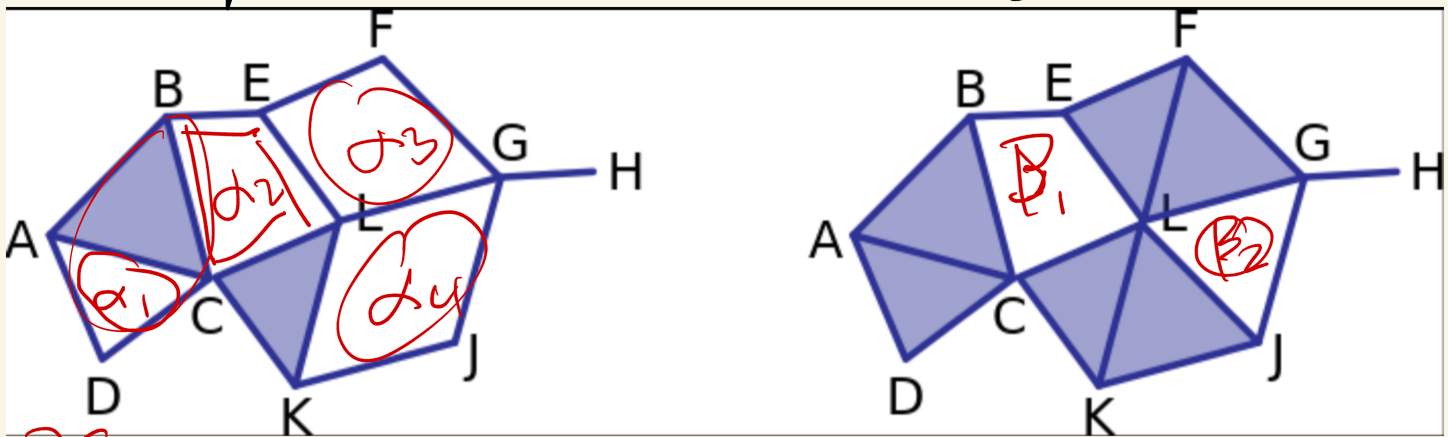
Example:



$$f_*: \quad \begin{array}{cc} & \alpha \\ \begin{array}{c} \beta \\ \gamma \end{array} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}$$

Another!

$$K \hookrightarrow L$$



α_1

$\hookrightarrow \alpha_1' = AD + DC + AB + BC$

$H_1(K)$ generated by: $[\alpha_1] = [AC + AD + CD]$

$[\alpha_2] = [BC + BE + CL + EL]$

$[\alpha_3] = [EF + EL + FG + GL]$

$[\alpha_4] = [GJ + GL + JK + KL]$

$\cong \mathbb{Z}_2^4$

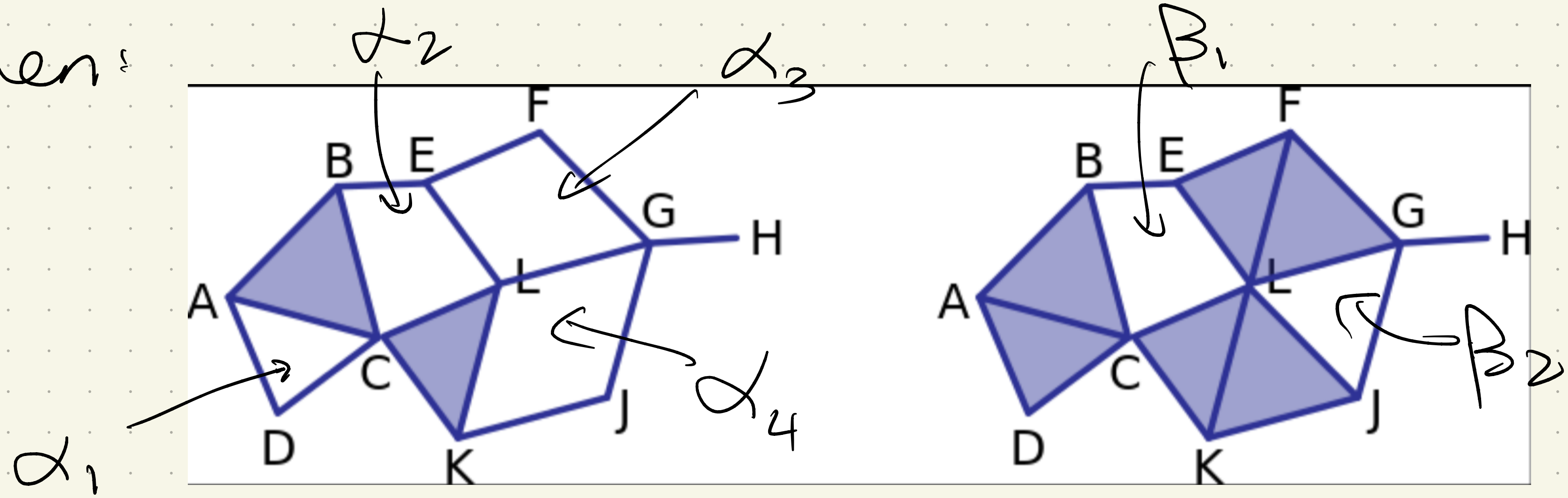
& $H_1(L)$ by $[\beta_1] = [BC + BE + CL + EL]$

$[\beta_2] = [GL + GJ + LJ]$

$\cong \mathbb{Z}^2$

$\longrightarrow$

Then:



$$\begin{array}{c c} & \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Relative Homology

Idea: compute homology of a complex relative to a subcomplex $L \subseteq K$

Take L a subcomplex of K.

$\Rightarrow$ $C_p(L)$ is a subgroup of $C_p(K)$.

Quotients again!

$$C_p(K)/C_p(L) = C_p(K, L)$$

Relative chain group

Boundaries extend naturally:

$$\partial_p^{K,L} : [c_p] \longrightarrow [\partial_p c_p]$$

Can check all the same things:

$$\partial_{p-1}^{K,L} \partial_p^{K,L} = 0$$

so $Z_p(K,L) = \ker \partial_p^{K,L}$

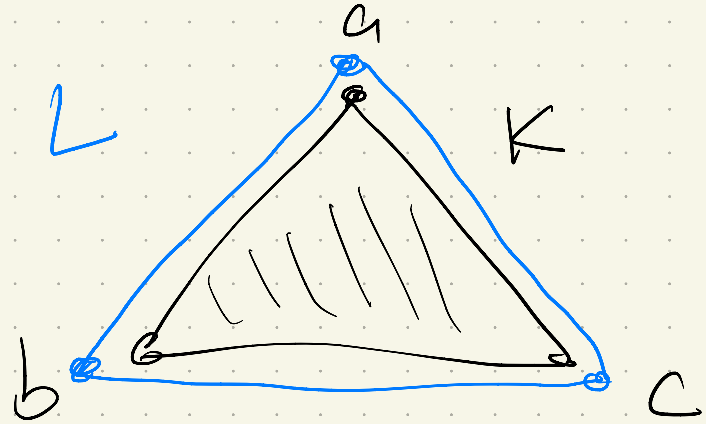
$B_p(K,L) = \operatorname{im} \partial_{p+1}^{K,L}$

$$\text{\& } H_p(K,L) = Z_p(K,L) \big/ B_p(K,L)$$

But why?

Essentially, equivalent to "coning off" L, so L has no topology.

Example:



$$C_2(K) = \langle 0, [a_0 a_1 a_2] \rangle$$

$$C_2(L) = 0$$

$$\Longrightarrow C_2(K, L) = \langle 0, [a_0 a_1 a_2] \rangle$$

$$C_1(K) = \langle 0, [ab], [ac], [bc] \rangle$$

$$C_1(L) = \langle 0, [ab], [ac], [bc] \rangle$$

$$\Longrightarrow C_1(K, L) = \langle 0 \rangle$$

$$\& \quad C_0(K) = C_0(L) = \langle 0, [a], [b], [c] \rangle$$

$$\Longrightarrow C_0(K, L) = 0$$

Then:

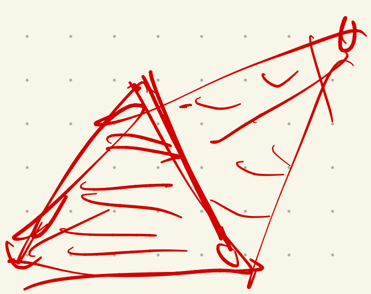
$$0 \xrightarrow{\partial_3} C_2(K,L) \xrightarrow{\partial_2} C_1(K,L) \xrightarrow{\partial_1} C_0(K,L)$$

$C_2(K,L) = \mathbb{Z}_2$ (generated by $[abc]$), $C_1(K,L) = 0$, $C_0(K,L) = 0$

So $H_1(K,L)$ + $H_0(K,L)$ are both 0.

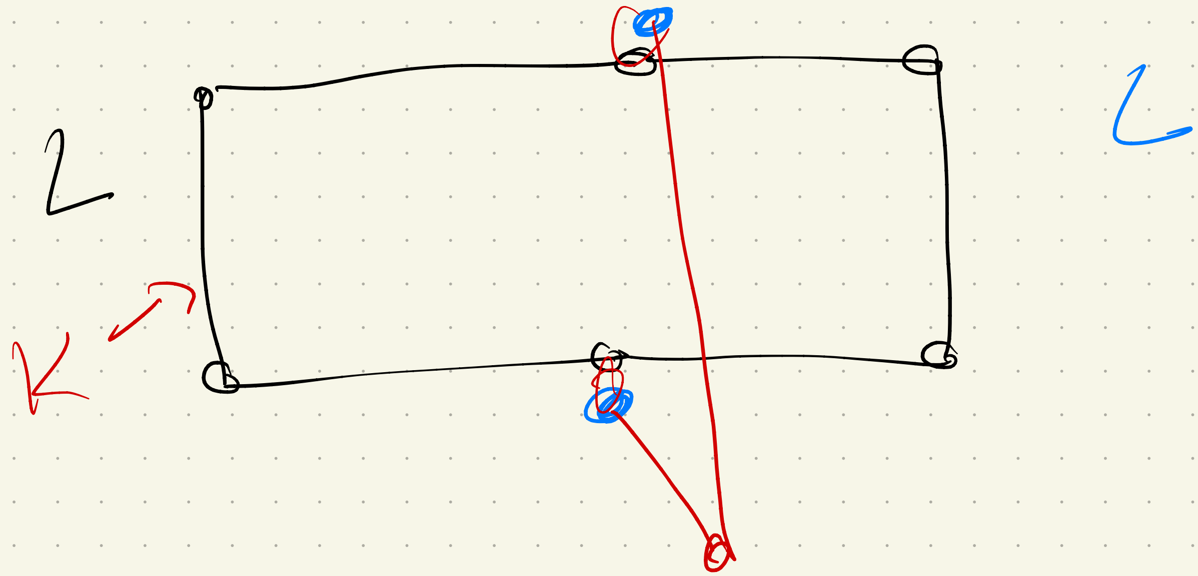
$$H_2(K,L) = \ker \partial_2 / \operatorname{im} \partial_3$$

$$= \langle \{abc\} \rangle / 0 = \mathbb{Z}_2$$



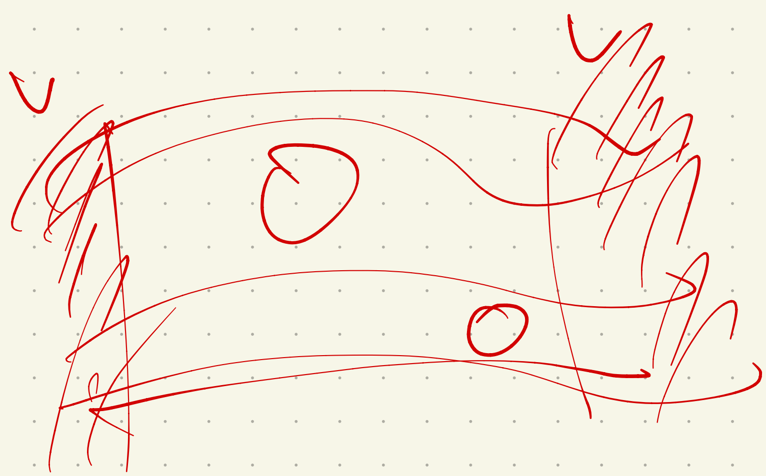
Faster:

"cone off" L



$$H_1(K) = \mathbb{Z}_2$$

$$H_1(K, L) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_2^2$$



Book also covers singular homology,
as well as cohomology.

I'm skipping these for now, but
we might revisit...

Next time: filtrations, &
using this all for persistent homology